### Introduction

Let \( f_1, \ldots, f_n \in k[x_1, \ldots, x_n] \) such that \( V(f_1, \ldots, f_n) \subseteq k[x_1, \ldots, x_n] \) is zero-dimensional. The intersection multiplicity \((p, f_1, \ldots, f_n)\) at the point \( p \in V(f_1, \ldots, f_n) \) specifies the weights of the weighted sum in Bézout's Theorem.

The number \( I(p, f_1, \ldots, f_n) \) is not natively computable by Maple while it is computable by Singular and Magma—but only when all coordinates of \( p \) are in \( k \).

We are interested in removing this algorithmic limitation. We combine Fulton's Algorithm and the theory of regular chains, leading to a complete algorithm for \( n = 2 \). Moreover, we propose algorithmic criteria for reducing the case of \( n > 2 \) variables to the bivariate one. Experimental results are reported.

### The case of two plane curves

Intuitively, the intersection multiplicity (IM) of two plane curves at a given point counts the number of times that these curves intersect at that point. More formally, given an arbitrary field \( k \) and two bivariate polynomials \( f, g \in k[x, y] \), consider the affine algebraic curves \( C = \{ f = 0 \} \) and \( D = \{ g = 0 \} \) in \( k^2 = \mathbb{R}^2 \), where \( \mathbb{R} \) is the algebraic closure of \( k \). Let \( p \) be a point in the intersection.

The intersection multiplicity of \( p \) in \( V(f, g) \) is defined to be

\[
I(p, f, g) = \dim_k \mathcal{O}_{\mathbb{A}^2, p}/(f, g)
\]

where \( \mathcal{O}_{\mathbb{A}^2, p} \) and \( \mathcal{O}_{\mathbb{A}^2, p}/(f, g) \) are the local ring at \( p \) and the dimension of the vector space \( \mathcal{O}_{\mathbb{A}^2, p}/(f, g) \).

Remarkably, and as pointed out by Fulton in his Intersection Theory, the intersection multiplicities of the plane curves \( C \) and \( D \) satisfy a series of properties which uniquely define \( I(p, f, g) \) at each point \( p \in V(f, g) \). Moreover, the proof of this remarkable fact is constructive, which leads to an algorithm.

#### Fulton's Properties

1. \( I(p, f, g) \) is a non-negative integer for any \( C \), \( D \), and \( p \) such that \( C \) and \( D \) have no common component at \( p \). We set \( I(p, f, g) = \infty \) if \( C \) and \( D \) have a common component at \( p \).
2. \( I(p, f, g) = 0 \) if and only if \( p \in C \cap D \).
3. \( I(p, f, g) \) is invariant under affine change of coordinates on \( \mathbb{A}^2 \).
4. \( I(p, f, g) = I(p, f, f/g) \).

#### Fulton's Algorithm

**Input:** \( p = (\alpha, \beta) \in \mathbb{A}^2(k) \) and \( f, g \in \mathcal{O}_{\mathbb{A}^2, p} \).

**Output:** \( I(p, f, g) \in \mathbb{N} \) satisfying (2)-(7) below.

If \( f(p) \neq 0 \) or \( p \neq 0 \) then

1. Return \( \deg f/\deg g \).
2. \( \deg g(\alpha, \beta) \).
3. Assume \( \alpha > x \).
4. Write \( f(y - \beta) \cdot h \) and \( g(y, \beta) = (y - \alpha)^m (a_0 + a_1 (y - \alpha) + \cdots) \).
5. Return \( m \).
6. \( \deg f/\deg g \).
7. Return \( 1 \).

On the other hand, at any point of \( V(f_1, f_2) \), we use the techniques developed before.

When this reduction does not apply, a priori, we can look for a more favorable system of generators.

For instance, consider the system \( O_{\mathbb{A}^2} \):

\[
x^2 + y^2 + z - 1 = 0 \quad \text{and} \quad y^2 + z^2 - 1 = 0.
\]

The above theorem does not apply. However, if one uses the first equation, say \( x^2 + y^2 + z - 1 = 0 \) to eliminate \( z \) from the two other, we obtain two bivariate polynomials \( f, g \in k[x, y] \). At any point of \( V(h, f, g) \) the tangent cone of the curve \( V(f, g) \) is independent of \( z \); in some sense it is “vertical”. On the other hand, at any point of \( V(h, f, g) \), the tangent space of \( V(h) \) is not vertical. Thus, the previous theorem applies without computing any tangent cones.

#### Reducing the \( n \)-dimensional case to the \( n - 1 \) case

The intersection multiplicity of \( p \) in \( V(f_1, \ldots, f_n) \) is given by

\[
I(p, f_1, \ldots, f_n) = \dim_k \mathcal{O}_{\mathbb{A}^n, p}/(f_1, \ldots, f_n)
\]

where \( \mathcal{O}_{\mathbb{A}^n, p} \) and \( \mathcal{O}_{\mathbb{A}^n, p}/(f_1, \ldots, f_n) \) are respectively the local ring at the point \( p \) and the dimension of the vector space \( \mathcal{O}_{\mathbb{A}^n, p}/(f_1, \ldots, f_n) \).

The next theorem reduces the \( n \)-dimensional case to \( n - 1 \) under assumptions which state that \( f_n \) does not contribute to \( I(p, f_1, \ldots, f_n) \).

**Theorem 2:** Assume that \( h_n \in V(f_1, \ldots, f_{n-1}) \) is non-singular at \( p \). Let \( v_n \) be its tangent hyperplane at \( p \). Assume that \( h_n \) meets each component (through \( p \)) of the curve \( C = \{ f_1 = \cdots = f_{n-1} = 0 \} \) transversely (that is, the tangent cone \( T_{\mathbb{A}^n, p}(C) \) intersects \( v_n \) only at the point \( p \)). Let \( h \in k[x_1, \ldots, x_{n-1}] \) be the degree 1 polynomial defining \( v_n \). Then, we have \( I(p, f_1, \ldots, f_{n-1}) = I(p, f_1, \ldots, f_{n-1}, h) \).

#### Reduction in practice

How to use this theorem in practice? Assume that the coefficient of \( x_n \) in \( h_n \) is non-zero, thus \( h_n = x_n + h' \), where \( h' \in k[x_1, \ldots, x_{n-1}] \). Hence, we can rewrite the ideal \( (f_1, \ldots, f_{n-1}, h') \) as \( (g_1, \ldots, g_{n-1}, h) \) where \( g_i \) is obtained from \( f_i \) by substituting \( x_n \) with \( h' \). If instead of a point \( p \), we have a zero-dimensional regular chain \( T \subset k[x_1, \ldots, x_{n-1}] \), we use the techniques developed before.

**Experimental Results**

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- **On Fulton’s Algorithm for Computing Intersection Multiplicities**
- Steffen Marcus, Marc Moreno Maza, and Paul Vrbik