What Is Homotopy Continuation?

Homotopy continuation, like Newton’s method, is an iterative approach for finding the isolated complex roots of a polynomial. But unlike Newton’s method this process guarantees every isolated root will be found if seeded by a finite number of appropriately chosen starting points. The underlying idea is:

1. Connect the solutions of an exactly solvable system, \( q(z) = 0 \), with the solutions of the desired (target) system, \( p(z) = 0 \):

\[
H(z, t) = tq(z) + (1 - t)p(z).
\]

\( H(z, 1) = q(z) \) is the start system, e.g. \( z^d - 1 \).

\( H(z, 0) = p(z) \) is the target system.

2. Track the solution path as \( t \) goes from 1 to 0, using:

\[
\frac{dH}{dt} = H_z \frac{dz}{dt} + H_t = 0.
\]

Tracking:

For basic prediction and correction, consider the first order taylor approximation.

\[
H(z + \Delta z, t + \Delta t) = H_z(t, 1) \Delta z + H_t(t, 1) \Delta t
\]

One example where tracking fails.

\[-7z^8 + 22z^4 - 55z^3 - 94z^2 + 87z - 56.
\]

The brown and green paths converge to the real root -1.6 whereas the blue and yellow paths converge to the complex root 0.4-0.5i. The red root is escaping to infinity and is (quickly) flagged as a failed path. We do not find all roots.

One can guarantee (with probability 1) that we find all roots.

Basins of Attraction

At the correction step, Newton’s method can throw the path of course by converging to the wrong root. Unfortunately it is hard to predict if this is going to happen.

Newton’s method will converge to different roots depending on what initial value it is seeded.

Basin of attraction of a root

\[ \{ \text{initial points yielding the root} \} \]

To visualize this we again use stereographic projection to plot the paths on a Riemann sphere so we may see paths converge to infinity (the north pole).

The target system \( p(z) \) may have up to 16 roots so we must track 16 paths. This is illustrated to the left. All possible pairs \((x_i, y_j)\) with \(0 \leq i, j \leq 3\) constitute our 16 start points. Each sphere represents a path that a single component of the solution \((x, y)\) takes. We observe that half the paths diverge.

References
