MAT223 Lecture Notes

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Contents

Linear algebra is one of the best studied and understood fields in mathematics. The amount of attention it receives is warranted by the shear extent of its applicability, in both pure mathematics and applied mathematics, physics, computer science, engineering, etc. In a very broad sense, it studies linear systems of equations, vector spaces, and linear maps between vector spaces. This course will introduce you to the fundamentals of linear algebra, with a focus on low-dimensional spaces such as $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, though we will cover $\mathbb{R}^n$ towards the end.

I have heard linear algebra earnestly proclaimed as "the single most useful mathematics you will learn as an undergraduate." While some people might dissent, it's hard to overstate the utility of linear algebra.

1 Linear Systems

You've likely seen examples of linear systems before. For example, you might have been asked to find a solution to

$$\begin{aligned} 2x - 3y &= -7 \\ -x + 2y &= 5. \end{aligned}$$

This is not too difficult with only two equations and two unknowns, but what if we add more equations and more unknowns? Something along the lines of

$$\begin{aligned} x + y + z &= 4 \\ x + 2y + 3z &= 9 \\ 2x + 3y + z &= 7 \end{aligned}$$

is much more difficult. You can imagine we could make this four equations, five equations, etc. We will develop a scheme for solving these types of systems.

1.1 Linear Equations and Systems

Generally speaking, the word *linear* means something that respects additions and multiplication by real numbers. For example, the function $f(x) = 2x$ is linear, since

$$f(x+y) = 2(x+y) = f(x) + f(y), \qquad \text{and} \qquad f(cx) = 2cx = cf(x).$$

The word linear is used because the graph of f is precisely a straight line in the plane. Linear things play so nicely with addition and multiplication. Be careful not to confuse this with the analytical notion of linear equations, $y = mx + b$, which are often referred to instead as *affine equations*. The difference between affine and linear equations is precisely the translation component.

A *linear equation* is any equation of the form

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$$

for $c_1, c_2, \ldots, c_n, b \in \mathbb{R}$. We refer to the c_i as *coefficients* of the linear equation, and b as the *constant term*. When $n = 2$ this becomes the equation

$$c_1x_1 + c_2x_2 = b,$$

and the collection of x_1 and x_2 which satisfy this equation again forms a line in the plane. For example,

$$2x_1 - 3x_2 = -7$$

looks like the line given in Figure 1.

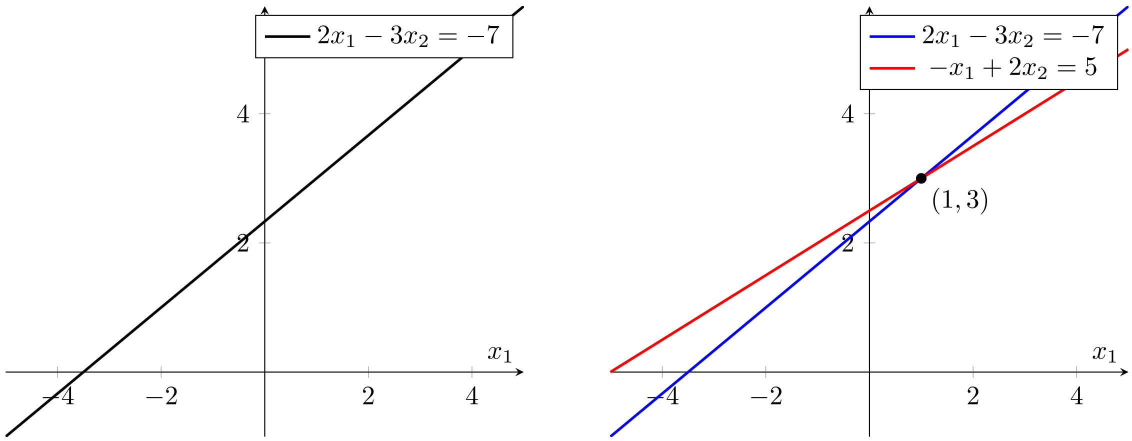

Figure 1: Left: The solutions to $2x_1 - 3x_2 = -7$ form a line in the plane. Right: There is a single solution to the system (1.1), which is the point where the two solution sets intersect.

A *linear system of equations* is a finite collection of linear equations:

$$\begin{array}{ccccccc} c_{1,1}x_1 + & c_{1,2}x_2 + & \cdots + & c_{1,n}x_n = & b_1 \\ c_{2,1}x_1 + & c_{2,2}x_2 + & \cdots + & c_{2,n}x_n = & b_2 \\ \vdots & \vdots & & \vdots & \\ c_{m,1}x_1 + & c_{m,2}x_2 + & \cdots + & c_{m,n}x_n = & b_m \end{array}.$$

This particular system has m equations in n unknowns. A solution to this system is any collection of n numbers $s_1, s_2, \ldots s_n$ such that

$$\begin{array}{ccccccc} c_{1,1}s_1 + & c_{1,2}s_2 + & \cdots + & c_{1,n}s_n = & b_1 \\ c_{2,1}s_1 + & c_{2,2}s_2 + & \cdots + & c_{2,n}s_n = & b_2 \\ \vdots & \vdots & & \vdots & \\ c_{m,1}s_1 + & c_{m,2}s_2 + & \cdots + & c_{m,n}s_n = & b_m \end{array};$$

that is, each equation is satisfied by the $s_1, \ldots, s_n$ *simultaneously*. For example, consider the system

$$\begin{array}{r} 2x_1 - 3x_2 = -7 \\ -x_1 + 2x_2 = 5 \end{array}. \tag{1.1}$$

The point $(-2, 1)$ satisfies the first equation, since

$$2(-2) - 3(1) = -4 - 3 = -7,$$

but this does not satisfy the second equation, as

$$-(-2) + 2(1) = 4 + 2 = 6 \neq 5.$$

In fact, the only simultaneous solution to both equations is $(x_1, x_2) = (1, 3)$.

Geometrically, the solutions to a single equation $ax + by = c$ form a line in the plane – there are infinitely many such solutions. A point in the plane is simultaneously a solution to two such equations if it occurs at their intersection, as in Figure 1. This immediately implies there are only three possible cases:

- There are no solutions: The lines are parallel.
- There is one unique solution: The lines cross at one point.
- There are infinitely many solutions: The lines overlap.

This same idea will extend beyond two dimensions, as we'll see in Section 1.3.2.

Definition 1.1

A linear system of equations is *consistent* if it admits at least one solution, and *inconsistent* otherwise.

The special case where all the constant terms are zero is afforded a special name.

Definition 1.2

A linear system is *homogeneous* if all its constant terms are zero.

The following linear system is homogeneous,

$$\begin{aligned} -3x_1 + 7x_2 - 10x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0. \\ -x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Notice that $(x_1, x_2, \ldots, x_{n-1}, x_n) = (0, 0, \ldots, 0, 0)$ is always a solution to a homogeneous system, called the *trivial solution*. Hence homogeneous systems are always consistent. While the trivial solution is important, we're often more interested in non-trivial solutions.

1.2 Parameterizations of Solutions

Solutions to linear systems come in three flavours: There are either no solutions, a single unique solution, or infinitely many solutions. We'll later develop criteria for determining the cardinality of a solution set without explicitly finding the solutions. For now, let's analyze how infinitely many solutions can be conveyed in a finite manner.

The equation $2x_1 - 3x_2 = -7$ has infinitely many solutions. Indeed, by solving for x_2 we can write $x_2 = (2x_1 + 7)/3$. By allowing x_1 to vary, we get the collection of all possible solutions. For example, setting $x_1 = 0$ gives $x_2 = 7/3$, while $x_1 = 1$ gives $x_2 = 3$. There is no restriction on the parameter x_1, but to differentiate the equation with the solution set, we introduce a letter in lieu of x_1. Taking t as our *parameter*, we set $x_1 = t$ and the solutions to the linear equation are

$$(x_1, x_2) = \left(t, \frac{2t + 7}{3}\right) \quad \text{for} \quad t \in \mathbb{R}.$$

In fact, we can verify this by substituting this back into our original equation:

$$2x_1 - 3x_2 = 2t - 3\left(\frac{2t+7}{3}\right) = 2t - (2t+7) = -7.$$

Example 1.3

Show that

$$(x_1, x_2, x_3, x_4) = (1 - 2s + 4t, 3s - t, s, t)$$

is a solution to the linear system

$$\begin{aligned} x_1 + \quad\quad 2x_3 - 4x_4 &= 1 \\ x_2 - 3x_3 + \quad x_4 &= 0 \end{aligned} \tag{1.2}$$

for any $s, t \in \mathbb{R}$.

Solution. By substituting $(x_1, x_2, x_3, x_4) = (1 - 2s + 4t, 3s - t, s, t)$ into our system, we get

$$\begin{aligned} x_1 + 2x_3 - 4x_4 &= (1 - 2s + 4t) + 2s - 4t = 1 \\ x_2 - 3x_3 + x_4 &= (3s - t) - 3s + t = 0 \end{aligned}$$

which is what we wanted to show. ∎

Our objective is usually to find the parameterization of an infinite solution set. In the case of Example 1.3 we solve the first equation in (1.2) for x_1, and the second equation for x_2, to get

$$\begin{aligned} x_1 &= 1 - 2x_3 + 4x_4 \\ x_2 &= 3x_3 - x_4. \end{aligned}$$

We cannot possibly solve for any more variables, so let x_3 and x_4 be parameters, say s and t respectively. Expressing x_1 and x_2 in terms of s and t gives a two-parameter solution set

$$(x_1, x_2, x_3, x_4) = (1 - 2s + 4t, 3s - t, s, t),$$

which agrees with what we found above. Of course, this is not the only parameterization of the solution, as we could have solved for other variables as well.

1.3 Matrix Representations of Linear Systems

An $m \times n$ matrix is collection of nm numbers, arranged into m rows and n columns. For example,

$$\begin{bmatrix} 1/2 & -\pi & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

is a 2×3 matrix of real numbers. Given a linear system

$$\begin{array}{ccccccc} c_{1,1}s_1 & + & c_{1,2}s_2 & + \cdots + & c_{1,n}s_n & = & b_1 \\ c_{2,1}s_1 & + & c_{2,2}s_2 & + \cdots + & c_{2,n}s_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \\ c_{m,1}s_1 & + & c_{m,2}s_2 & + \cdots + & c_{m,n}s_n & = & b_m \end{array};$$

we encode this information in an (augmented) $m \times (n+1)$ matrix whose entries are the coefficients and constant terms:

$$\left[\begin{array}{cccc|c} c_{1,1} & c_{1,2} & \cdots & c_{1,n} & b_1 \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,n} & b_m \end{array}\right].$$

For example, the linear systems we've thus far seen are encoded as follows:

$$\begin{array}{cc} \begin{aligned} 2x - 3y &= -7 \\ -x + 2y &= 5 \end{aligned} & \left[\begin{array}{cc|c} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right] \\[1em] \begin{aligned} x + y + z &= 4 \\ x + 2y + 3z &= 9 \\ 2x + 3y + z &= 7 \end{aligned} & \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 9 \\ 2 & 3 & 1 & 7 \end{array}\right] \\[1em] \begin{aligned} x_1 + 2x_3 - 4x_4 &= 1 \\ x_2 - 3x_3 + x_4 &= 0 \end{aligned} & \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 1 \\ 0 & 1 & -3 & 1 & 0 \end{array}\right] \end{array}.$$

Now let's think about what operations we can do to our system of equations while preserving the solutions, and see how those operations translate to the matrix picture.

1. **We can interchange any two equations.** Certainly it does not matter whether we solve the system

$$\begin{aligned} 2x - 3y &= -7 \\ -x + 2y &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} -x + 2y &= 5 \\ 2x - 3y &= -7 \end{aligned}$$

   so we can interchange the rows of a matrix,

$$\left[\begin{array}{cc|c} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right] \xrightarrow{\mathrm{R}_1 \leftrightarrow \mathrm{R}_2} \left[\begin{array}{cc|c} -1 & 2 & 5 \\ 2 & -3 & -7 \end{array}\right]$$

2. **We can multiply a row by a non-zero number.** For example, if s_1, s_2 satisfy

$$2s_1 - 3s_2 = -7$$

   then multiplying everything by 5 gives

$$10s_1 - 15s_2 = -35.$$

   So long as the coefficients *and* the constant term are both multiplied by the same constant, (s_1, s_2) is still a solution.

$$\left[\begin{array}{cc|c} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right] \xrightarrow{5\mathrm{R}_1 \to \mathrm{R}_1} \left[\begin{array}{cc|c} 10 & -15 & -35 \\ -1 & 2 & 5 \end{array}\right]$$

3. **We can add a multiple of one row to another.** For example, if (s_1, s_2) is a solution to

$$\begin{aligned} 2s_1 - 3s_2 &= -7 \\ -s_1 + 2s_2 &= 5 \end{aligned}$$

then taking 3 times the first row and adding it to the second gives

$$\begin{array}{cccccccc} & 3(& 2s_1 & - & 3s_2 & = & -7) \\ + & & -s_1 & + & 2s_2 & = & 5 \\ \hline & & 5s_1 & - & 7s_2 & = & -16. \end{array}$$

At the matrix level, we get

$$\left[\begin{array}{rr|r} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right] \xrightarrow{3\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{rr|r} 2 & -3 & -7 \\ 5 & -7 & -16 \end{array}\right]$$

These are called *elementary row operations* (EROs).

How does this help us solve linear systems? At the moment, these matrices represent a notation for convenient bookkeeping, so let's compare this to how we would normally solve this system. Take the system

$$\begin{aligned} 2x_1 - 3x_2 &= -7 \\ -x_1 + 2x_2 &= \ \ 5. \end{aligned}$$

We will add 2 times the second row to the first and add them together to get

$$\begin{array}{cccccccc} & & 2x_1 & - & 3x_2 & = & -7 \\ + & 2(& -x_1 & + & 2x_2 & = & 5) \\ \hline & & & & x_2 & = & 3. \end{array}$$

Hence $x_2 = 3$. We can substitute this back into the equation $2x_1 - 3x_2 = -7$ to get

$$2x_1 - 3(3) = -7 \qquad \Rightarrow \qquad 2x_1 = 2 \qquad \Rightarrow \qquad x_1 = 1$$

and we get the solution $(x_1, x_2) = (1, 3)$. The objective here is *variable elimination*; that is, to use the elementary row operations to remove as many of the variables as possible. At the matrix level, we will do something similar. We will perform our elementary row operations to transform our matrix to *row-echelon form* (REF), in which

1. Any rows consisting of purely zeros occurs at the bottom of the matrix,
2. The first non-zero entry of any row is a 1, called the *leading* 1,
3. Each leading 1 occurs to the right of any leading 1 above it.

For example, the following matrix is in row-echelon form:

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.3}$$

Why is this useful? Think about the corresponding linear system of an augmented REF matrix. It might look something like

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right], \qquad \text{which becomes} \qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= \ \ 4 \\ x_3 &= -1. \end{aligned}$$

In row-echelon form, the x_1-dependency has been removed from all equations below the first. The same is true for x_3, though trivially so. In this form it is easy to solve for x_3 and x_1, using x_2 as a free parameter. So for example, the solution to this linear system is

$$\begin{aligned} x_3 &= -1 \\ x_2 &= \text{free parameter } s \\ x_1 &= 4 - 2x_2 - 3x_3 = 7 - 2x_2 \end{aligned} \qquad \Rightarrow \qquad (x_1, x_2, x_3) = (7 - 2s, s, -1).$$

By eliminating as many variables as possible, we refined the system into something which is easier to interpret and solve.

Example 1.4

Perform elementary row operations on the augmented matrix

$$\left[\begin{array}{cc|c} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right]$$

to turn it into row-echelon form.

Solution. Using our elementary row operations, we have the following

$$\left[\begin{array}{cc|c} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} -1 & 2 & 5 \\ 2 & -3 & -7 \end{array}\right] \xrightarrow{(-1)R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -2 & -5 \\ 2 & -3 & -7 \end{array}\right]$$
$$\xrightarrow{(-2)R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & -5 \\ 0 & 1 & 3 \end{array}\right].$$

Were we to convert this back into its corresponding linear system, we would have

$$\begin{aligned} x_1 - 2x_2 &= -5 \\ x_2 &= \ \ 3. \end{aligned}$$

Knowing that $x_2 = 3$ we can solve for $x_1 = -5 + 2x_2 = 1$, which is the same solution we got earlier. ∎

We say that a matrix is in *reduced row-echelon form* (RREF) if each leading 1 is also the only non-zero entry in its column, so if every $*$ entry in (1.3) is a zero, that matrix is in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The linear system corresponding to an augmented matrix in RREF has been solved as much as possible: it's no longer necessary to do any back substitution, you need only read off the variable solutions.

Example 1.5

Turn the matrix

$$\left[\begin{array}{cc|c} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right]$$

into reduced row-echelon form.

Solution. The key is to first put our matrix into row-echelon form, then "work our way back upwards." From Example 1.4 we know the REF form, so

$$\left[\begin{array}{cc|c} 2 & -3 & -7 \\ -1 & 2 & 5 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{cc|c} 1 & -2 & -5 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{2R_2+R_1\to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right].$$

The corresponding linear system is just the solution $(x_1, x_2) = (1, 3)$. ■

1.3.1 Gaussian Elimination

This process of turning a matrix into row-echelon form, and eventually into reduced row-echelon form, is called *Gaussian elimination*. The algorithm for converting a matrix to row-echelon form is as follows:

1. If your matrix consists entirely of zeros, stop.
2. Find the first column with a non-zero entry and move it to the top.
3. Normalize the row to create a leading 1.
4. Add multiples of this row to the rows below it, so that the elements under the leading 1 become 0.

To solve a linear system, take the corresponding augmented matrix and convert it to row-echelon form. If there is a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$ then your system has no solutions. Otherwise, you can begin the process of backwards substitution, starting at the bottom, to solve your linear system.

Another option is to turn your augmented matrix into row-echelon form, and use a similar procedure to Gaussian elimination to turn elements above the leading 1's into zero as well. This will give you the reduced row-echelon form.

Example 1.6

Use Gaussian elimination to turn the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 9 \\ 2 & 3 & 1 & 7 \end{array}\right]$$

into row-echelon form. Use backwards substitution to solve the system. By turning the matrix into reduced row-echelon form, confirm your answer.

Solution. Applying Gaussian elimination, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 9 \\ 2 & 3 & 1 & 7 \end{array}\right] \xrightarrow[(-2)\mathrm{R}_1+\mathrm{R}_3\to\mathrm{R}_3]{(-1)\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{(-1)\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -3 & -6 \end{array}\right]$$

$$\xrightarrow{(-1/3)\mathrm{R}_3\to\mathrm{R}_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding linear system is

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_2 + 2x_3 &= 5 \\ x_3 &= 2. \end{aligned}$$

Setting $x_3 = 3$ and substituting into the second equation gives $x_2 = 5 - 2x_3 = 1$. Substituting both values into the first equation gives

$$x_1 = 4 - x_2 - x_3 = 4 - (1) - (2) = 1$$

so our solution is $(x_1, x_2, x_3) = (1, 1, 2)$. You can check your answer by substituting this into the original linear system of equations.

To turn this into reduced row-echelon form, we start with our row-echelon form and work upwards:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[(-1)\mathrm{R}_1+\mathrm{R}_1\to\mathrm{R}_1]{(-2)\mathrm{R}_3+\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{(-1)\mathrm{R}_2+\mathrm{R}_1\to\mathrm{R}_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

which gives us the same solution above, $(x_1, x_2, x_3) = (1, 1, 2)$. ■

Example 1.7

Find the solution(s) to the linear system

$$\begin{aligned} x_1 + 2x_2 - 4x_3 &= 10 \\ 2x_1 - x_2 + 2x_3 &= 5 \\ x_1 + x_2 - 2x_3 &= 7 \end{aligned}$$

Solution. I'm going straight to RREF, but you are free to do backwards substitution if you like.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 3 & -4 & 1 & 5 \\ 1 & 1 & -2 & 7 \end{array}\right] \xrightarrow[(-1)\mathrm{R}_1+\mathrm{R}_3\to\mathrm{R}_3]{(-2)\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & -5 & 10 & -15 \\ 0 & -1 & 2 & -3 \end{array}\right] \xrightarrow{(-1/5)\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & -3 \end{array}\right]$$

$$\xrightarrow{\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-2)\mathrm{R}_2+\mathrm{R}_1\to\mathrm{R}_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We cannot quite read off the solutions immediately. Instead, notice that there is no leading one for the third column. This means that x_3 is a free parameter, say s. Rewriting this but solving for x_1 and x_2 gives

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 3 + 2s \\ x_3 &= s \end{aligned}$$

so our final solution is $(x_1, x_2, x_3) = (4, 3 + 2s, s)$ for any $s \in \mathbb{R}$. ■

Example 1.8

Convert the following matrix to reduced row-echelon form:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & 2 & 3 & -1 & 2 \\ 1 & -1 & 2 & 1 & 1 \end{array}\right].$$

Solution. Applying Gaussian elimination we get

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & 2 & 3 & -1 & 2 \\ 1 & -1 & 2 & 1 & 1 \end{array}\right] \xrightarrow[\substack{R_1+R_3\to R_3 \\ -1R_1+R_4\to R_4}]{R_1+R_2\to R_2} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 4 & -3 & 3 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow[R_3\leftrightarrow R_4]{R_2\leftrightarrow R_3} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & 1 & 4 & -3 & 3 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[(-1/7)R_4\to R_4]{-2R_3+R_4\to R_4} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -2 & 1 \\ 0 & 1 & 4 & -3 & 3 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\substack{3R_4+R_2\to R_2 \\ 2R_4+R_1\to R_1}]{-3R_4+R_3\to R_3} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-R_3+R_1\to R_1]{-4R_3+R_2\to R_2} \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$ ■

Example 1.9

Which of the following matrices are in RREF?

$$A = \left[\begin{array}{rrrr|r} 1 & 3 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad B = \left[\begin{array}{rrr|r} 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad C = \left[\begin{array}{rr|r} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Solution. The matrix A is not in RREF, since there is a leading one in the $(2, 2)$ position but it is not the only non-zero element in its column. Both B and C are in RREF however, as they satisfy all the conditions necessary. ■

1.3.2 The Rank of a Matrix

Definition 1.10

Let A be a matrix. The *rank* of A is the number of leading 1's in its row-echelon form.

For example, the rank of the matrix from Example 1.7 is 2, while the rank of the matrix from Example 1.8 is 4. The number of leading ones is the same in REF and in RREF, so you could use either. The fact that the rank is independent of how we perform the Gaussian elimination shouldn't be clear, so you'll have to take my word for it.

Example 1.11

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix}.$$

Solution. Putting this matrix into row echelon form gives

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix} \xrightarrow[(-2)\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3]{\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

After scaling the second row, this matrix has leadings ones in the first and second columns only, and thus has rank 2. ■

This is not a good definition of rank, for several reasons which are hard to elaborate upon right now. Instead, we introduce this concept so that we can talk about the number of solutions a system can have. Notice that Examples 1.6 and 1.7 both consist of three equations in three unknowns, but the former has a unique solution while the latter had infinitely many solutions. The difference arises because of the rank, though not in an obvious way. Effectively, here's how you can see the number of solutions you should get. Suppose you have an augmented $m \times (n+1)$ matrix A in reduced echelon form. Let r be the rank of this matrix if the constant terms are removed.

Number of Solutions

1. If a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$ appears, there are no solutions.
2. If $r < n$ then you have infinitely many solutions, consisting of $n - r$ parameters. The parameters are those columns which do **not** contain leading ones.
3. If $r = n$ then you have a unique solution.

Indeed, the matrices in Examples 1.6 and 1.7 are both 3×4, but Example 1.6 has rank 3 and a unique solution, while Example 1.7 has rank 2 and infinitely many solutions.

Recall that a homogeneous system (Definition 1.2) is one in which all the constants are identically zero. I mentioned previously that the trivial solution $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is always a solution to a homogeneous system, so homogeneous systems either have one solution (the trivial one) or infinitely many. According to our criteria above for determining the number of solutions to a system, a homogeneous system will exhibit non-trivial solutions when its rank r is less than the number of variables n.

Example 1.12

Find and parameterize the non-trivial solutions to the linear system

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 - 3x_4 &= 0. \\ -x_1 - x_2 + x_3 + 3x_4 &= 0 \end{aligned}$$

Solution. We begin by putting our system into RREF:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 2 & 1 & -3 & 0 \\ -1 & -1 & 1 & 3 & 0 \end{array}\right] \xrightarrow[\mathrm{R_1+R_3 \to R_3}]{-2\mathrm{R_1+R_2 \to R_2}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right] \xrightarrow[-1\mathrm{R_2 \to R_2}]{2\mathrm{R_1+R_3 \to R_3}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{-\mathrm{R_1+R_1 \to R_1}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Our system has rank $r = 2$ and $n = 4$ variables, so there are non-trivial solutions in $n - r = 2$ parameters. The parameters are those columns without leading ones, namely $x_2 = s$ and $x_4 = t$. If we rewrite this as a linear system, it becomes

$$\begin{aligned} x_1 + x_2 + - 2x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -x_2 + 2x_4 \\ x_3 &= -x_4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -s + 2t \\ x_3 &= -t \end{aligned}.$$

Thus the solution to this system of equations is $(x_1, x_2, x_3, x_4) = (-s + 2t, s, -t, t)$. Note that this includes the trivial solution when $s = t = 0$. ■

Example 1.13

Determine if each statement is true or false. Suppose you are given a non-trivial linear system consisting of m equations and n unknowns.

1. The rank r of the augmented matrix encoding the system can be no larger than $\min\{m, n\}$.
2. If $n > m \geq 1$, the system admits infinitely many solutions.
3. If $m > n$, the system admits no solutions and is inconsistent.

Solution. We justify our answers below:

1. This is true. The rank corresponds to the number of leading ones. The number of leading ones is restricted by whichever is fewer, the number of rows, or the number of columns.
2. This is true. The rank of the linear system will be at most m, meaning there are $n - m > 0$ parameters, and hence infinitely many solutions.
3. This is false. For example, the system represented by the augmented matrix
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{array}\right]$$
has rank 1, while $n = 2$. Thus the system has infinitely many solutions. ■

1.4 Matrix Operations

We will generally denote a matrix by a capital letter, for example A. We denote the (i, j)-element (ith row, jth column) of A as A_{ij} and write $[A_{ij}]$ to refer to the matrix made up of these entries. For example, if
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -2 & -4 & -6 & -8 \end{bmatrix}$$
then $A_{2,3} = 7$ and $A_{3,4} = -8$.

Two matrices are equal if they are the same size and have identical elements. More precisely, if A and B are both $m \times n$ matrices, then $A = B$ if and only if $A_{ij} = B_{ij}$ for every $1 \leq i \leq m, 1 \leq j \leq n$. We can add two matrices of the same size as well, by saying that $(A + B)_{ij} = A_{ij} + B_{ij}$. For example, if
$$A = \begin{bmatrix} -1 & 4 & 2 \\ 0 & -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ -2 & 4 & 3 \end{bmatrix}$$
then
$$A + B = \begin{bmatrix} -1 & 4 & 2 \\ 0 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ -2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 + (-1) & 4 + 4 & 2 + 6 \\ 0 + (-2) & -3 + 4 & 0 + 3 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 8 \\ -2 & 1 & 3 \end{bmatrix}.$$
We can perform what's called *scalar multiplication* by taking $c \in \mathbb{R}$, and defining cA to be $(cA)_{ij} = cA_{ij}$. For example, if $c = 3$ and A is as above, then
$$3A = 3 \begin{bmatrix} -1 & 4 & 2 \\ 0 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 12 & 6 \\ 0 & -6 & 0 \end{bmatrix}.$$

Theorem 1.14

If A, B, C are $m \times n$ matrices, with $r, s, t \in \mathbb{R}$ then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $0 + A = A$ (where 0 is the 0-matrix)
4. $A + (-A) = 0$ where $-A = -1A$.
5. $r(A + B) = rA + rB$
6. $(r + s)A = rA + sA$
7. $(rs)A = r(sA)$

We will see how to multiply matrices in the Section 1.4.4, but I warn you now that it is not as straightforward as addition.

Example 1.15

Suppose that A and B are matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} x+y & 0 \\ -4 & x-y \end{bmatrix},$$

and that $4A + 2B = 0$. Find x, y.

Solution. By definition:

$$4A + 2B = \begin{bmatrix} -4 & 0 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 2x+2y & 0 \\ -8 & 2x-2y \end{bmatrix} = \begin{bmatrix} 2x+2y-4 & 0 \\ 0 & 2x-2y+12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which means we need $2x+2y = 4$ and $2x-2y = -12$. We can solve this linear system by introducing a matrix and row reducing:

$$\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 2 & -2 & -12 \end{array}\right] \xrightarrow{(-1)\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{cc|c} 2 & 2 & 4 \\ 0 & -4 & -16 \end{array}\right] \xrightarrow[(-1/4)\mathrm{R}_2\to\mathrm{R}_2]{(1/2)\mathrm{R}_1\to\mathrm{R}_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 4 \end{array}\right]$$
$$\xrightarrow{(-1)\mathrm{R}_2+\mathrm{R}_1\to\mathrm{R}_1} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 4 \end{array}\right]$$

so $x = -2$ and $y = 4$. ■

1.4.1 The Transpose of a Matrix

Given an $m \times n$ matrix $A = [A_{ij}]$, its transpose is the $n \times m$ matrix derived by interchanging the rows and columns. We denote the transpose by A^T. Hence if

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

To avoid large, awkward gaps in these notes, I will sometimes use the transpose to denote column vectors, such as the 3×1 *column vector* $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$.

Theorem 1.16

If A, B are $m \times n$ matrices and $c \in \mathbb{R}$, then

1. $(A^T)^T = A$,
2. $(cA)^T = cA^T$,
3. $(A+B)^T = A^T + B^T$.

Example 1.17

We say that a square $n \times n$ matrix A is anti-symmetric if $A + A^T = 0$, with 0 the zero-matrix. The trace of a matrix is the sum of its diagonal terms; that is,

$$\operatorname{Tr}(A) = A_{1,1} + A_{2,2} + \cdots + A_{n,n}.$$

Show that that the trace of an anti-symmetric matrix is zero.

Solution. Suppose our matrix A has components $A_{i,j}$. When we take the transpose, the rows and columns interchange, so that $[A^T]_{i,j} = A_{j,i}$. But notice that the diagonal elements of a square matrix are fixed under transposition: the diagonal of the original matrix is still the diagonal of the transpose. Hence

$$[A + A^T]_{i,i} = A_{i,i} + A_{i,i} = 2A_{i,i} = 0,$$

showing that $A_{i,i} = 0$. Thus the trace is

$$\operatorname{Tr}(A) = A_{1,1} + A_{2,2} + \cdots + A_{n,n} = 0 + 0 + \cdots + 0 = 0. \qquad \blacksquare$$

Definition 1.18

A matrix A is said to be *symmetric* if $A = A^T$, and *anti-symmetric* if $A = -A^T$.

From the definition of a symmetric matrix, we can immediately deduce that A must be a square matrix; that is, it has the same number of rows as columns. For example, the matrix

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 4 \\ -2 & 4 & -7 \end{bmatrix} \text{ is symmetric, while } \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix} \text{ is anti-symmetric.}$$

Example 1.19

Show that the sum of two symmetric matrices is symmetric.

Solution. Suppose that A and B are symmetric, so that $A = A^T$ and $B = B^T$. By properties of the transpose, we know that

$$(A+B)^T = A^T + B^T = A + B$$

showing that $A + B$ is symmetric, as required. $\blacksquare$

It shouldn't be too hard to convince yourself that the proof above generalizes to any combination of matrices.

1.4.2 Column Vectors

Very special cases occur when one of the dimensions of the matrix is 1. For example, an $m \times 1$ matrix is known as a column vector (aka column, column matrix), and a $1 \times n$ matrix is known as a row vector. Just like a 1×1 matrix can be though of as a real number, $\begin{bmatrix} a \end{bmatrix} \in \mathbb{R}$, we can think of column and row vectors as being elements in a higher dimensional space.

Definition 1.20

If n is a positive integer, the set $\mathbb{R}^n$ is the collection of all n-tuples of real numbers.

So for example,

$$(-5, \pi, 1001) \in \mathbb{R}^3, \quad (0,0,1,0) \in \mathbb{R}^4, \quad \underbrace{(1,0,1,0,\cdots,1,0)}_{\text{20-times}} \in \mathbb{R}^{20}.$$

Strictly speaking, the way I've defined $\mathbb{R}^n$ above carries no orientation of the n-tuple: we do not care whether it is written as a row or a column. For this reason, I will often conflate the two, writing row or column vectors as $\mathbf{x} \in \mathbb{R}$.

Elements in $\mathbb{R}^n$ can be thought of as either points, or arrows. For example, $(a,b) \in \mathbb{R}^2$ is either the point whose coordinates are (a,b), or the arrow pointing from the origin $(0,0)$ to (a,b). This is illustrated in Figure 2. In particular, if we think of them as arrows then we can add them together or multiply them by scalars:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

These are just the addition and scalar multiplication of matrices, but they can now be interpreted in a geometric way.

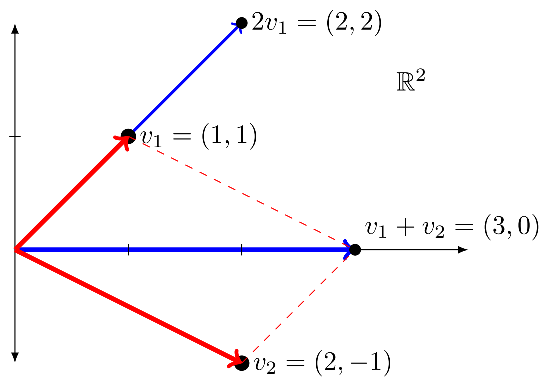

Figure 2: One may think of a vector as either representing a point in the plane (represented by the black dots) or as direction with magnitude (represented by the red arrows). The blue arrows correspond to the sum $v_1 + v_2$ and the scalar multiple $2v_1$. Notice that both are computed pointwise.

We haven't learned what a vector is yet, so I might be getting a bit ahead of myself, but for the moment take this to just be a matter of nomenclature. We will denote row and column vectors using bold font, such as $\mathbf{x}$. Since these are just special types of matrices, everything from Theorem 1.14 holds, such as

$$\begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 4 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ -7 \\ 5 \end{bmatrix}.$$

Given column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and scalars $c_1, c_2, \ldots, c_n$, we call anything of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

a *linear combination* of these vectors.

Column vectors in particular really come to play when looking at systems of linear equations. Consider the linear system

$$\begin{aligned} x_1 + 2x_2 - 4x_3 &= 10 \\ 2x_1 - x_2 + 2x_3 &= 5. \\ x_1 + x_2 - 2x_3 &= 7 \end{aligned}$$

Define column vectors whose elements are the coefficients of each x_i

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 5 \\ 7 \end{bmatrix}.$$

Thinking of the x_i as scalars, our linear system above is equivalent to

$$\begin{aligned} \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \mathbf{a}_3x_3 = \mathbf{b} \quad &\Leftrightarrow \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix} x_3 = \begin{bmatrix} 10 \\ 5 \\ 7 \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} x_1 + 2x_2 - 4x_3 \\ 2x_1 - x_2 + 2x_3 \\ x_1 + x_2 - 2x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 7 \end{bmatrix}, \end{aligned}$$

where in the last line, we use the fact that these column vectors are equal if and only if their components are equal. Hence linear systems are the same thing as linear combinations of column vectors.

Example 1.21

Determine whether $\mathbf{v}$ can be written as a linear combination of $\mathbf{x}, \mathbf{y}, \mathbf{z}$, where

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}.$$

Solution. If such a solution exists, then there are x_1, x_2, x_3 such that $\mathbf{v} = x_1\mathbf{x} + x_2\mathbf{y} + x_3\mathbf{z}$, which is the same as solving the linear system

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ x_1 + x_3 &= 3 \\ -x_1 + x_2 + 2x_3 &= 4 \end{aligned} \qquad \text{with matrix} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 0 & 1 & 3 \\ -1 & 1 & 2 & 4 \end{array}\right]$$

Applying Guassian elimination gives

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 0 & 1 & 3 \\ -1 & 1 & 2 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 5 \\ -1 & 1 & 2 & 4 \end{array}\right] \xrightarrow[R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 3 & 7 \end{array}\right]$$
$$\xrightarrow{(-1)R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 4 & 8 \end{array}\right] \xrightarrow{(1/4)R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\xrightarrow[R_3+R_2\to R_2]{(-1)R_3+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

This does indeed have a solution, showing that $\mathbf{x}+\mathbf{y}+2\mathbf{z}=\mathbf{v}$, so $\mathbf{v}$ is a linear combination of $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$. ■

1.4.3 Revisiting Homogeneous Systems

Note that solutions of linear systems are also vectors. An interesting property of homogeneous systems is that linear combinations of solutions are still solutions. For example, suppose that $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ are solutions to

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots \mathbf{a}_n x_n = \mathbf{0},$$

and let $c_1, c_2 \in \mathbb{R}$, then

$$\begin{aligned} \mathbf{a}_1(c_1 s_1 + c_2 t_1) &+ \mathbf{a}_2(c_1 s_2 + c_2 t_2) + \cdots + \mathbf{a}_n(c_1 s_n + c_2 t_n) \\ &= c_1(\mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \cdots + \mathbf{a}_n s_n) + c_2(\mathbf{a}_1 t_1 + \mathbf{a}_2 t_2 + \cdots + \mathbf{a}_n t_n) \\ &= c_1 \mathbf{0} + c_2 \mathbf{0} = \mathbf{0}. \end{aligned}$$

In addition, as all homogeneous systems have at least the trivial solution, we recast the "Number of Solutions" theorem (page 13) as follows:

Theorem 1.22

If an unaugmented $m \times n$ system A with rank r describes the coefficient matrix of a linear homogeneous system, then

1. The system has exactly $n - r$ 'basic' solutions, one for each parameter;
2. Every solution is a unique linear combination of those basic solutions.

The basic solutions are fundamental to the homogeneous system, in the sense that all other solutions are a linear combination of the basic solutions. At a practical level, the basic solutions are coefficients of the parameters.

Example 1.23

Consider the linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ -x_1 - 2x_2 + 2x_3 + x_5 &= 0 \\ -x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 &= 0 \end{aligned}$$

Determine the basic solutions and give a formula for general solutions.

Solution. Writing this as an augmented matrix and row-reducing, we get

$$\left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 1 & 0 \\ -1 & -2 & 2 & 0 & 1 & 0 \\ -1 & -2 & 3 & 1 & 3 & 0 \end{array}\right] \xrightarrow[\mathrm{R}_1+\mathrm{R}_3\to\mathrm{R}_3]{\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2} \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow{(-2)\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3} \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\mathrm{R}_3+\mathrm{R}_1\to\mathrm{R}_1} \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here we have rank 2 and 5 variables, so we expect there to by $5-2=3$ basic solutions corresponding to the three parameters. The variables x_1 and x_3 have the leading ones, so let $x_2 = s$, $x_4 = t$, $x_5 = u$ and write

$$\begin{aligned} x_3 &= -x_4 - 2x_5 = -t - 2u \\ x_1 &= -2x_2 + x_3 - x_4 - x_5 \\ &= -2s + (-t-2u) - t - u \\ &= -2s - 2t - 3u. \end{aligned}$$

By factoring the s, t, u, we can write this as a linear combination of three vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-2t-3u \\ s \\ -t-2u \\ t \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}_1} s + \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{b}_2} t + \underbrace{\begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{b}_3} u.$$

The three vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are each a solution to the homogeneous system, and are the basic solutions. Every other solution to the system is a linear combination of these three. ■

1.4.4 Matrix Multiplication

Before looking at matrix multiplication, we first consider the dot product.

Definition 1.24

Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ be an $1 \times n$ row vector, and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$ be an $n \times 1$ column vector. The *dot product* (inner product) of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Remark 1.25

1. Strictly speaking, the dot product is always between two column vectors or two row vectors, $\mathbf{x}$ and $\mathbf{y}$. This operation of combining row and column vectors is really a very deep thing, and the fact that it is equivalent to the dot product is a theorem. Naturally, by applying the transpose we can turn these into row or column vectors, whichever we please.

2. The dot product has a nice geometric interpretation, but we cannot yet describe it until we know how to visualize column/row vectors.

Example 1.26

Compute the dot products of $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{y} \cdot \mathbf{z}$, where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Solution. Applying our formulas, we have

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= (1 \times 2) + (0 \times -5) + (1 \times 1) = 3 \\ \mathbf{y} \cdot \mathbf{z} &= (2 \times -1) + (-5 \times 0) + (1 \times 2) = 0. \end{aligned}$$

■

Given an $n \times k$ matrix A and a $k \times m$ matrix B, the product AB is an $n \times m$ matrix, whose (i, j) entry is the dot product of the ith row of A and the jth column of B; that is,

$$(AB)_{ij} = \sum_{r=1}^{k} A_{ik} B_{kj}.$$

Alternatively, let $\mathbf{r}_i$ be the i-th row of A (of which there are n), and let $\mathbf{c}_j$ be the j-th column of B (of which there are m). Notice that both $\mathbf{r}_i$ and $\mathbf{c}_j$ have k-entries, so we can take their dot product, and the matrix product AB is

$$AB = \begin{pmatrix} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_n \end{pmatrix} \left(\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_m \end{array} \right) = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_m \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{r}_n \cdot \mathbf{c}_1 & \mathbf{r}_n \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_n \cdot \mathbf{c}_m \end{bmatrix}.$$

Explicitly multiplying two 2×2 matrices $A = [A_{ij}], B = [B_{ij}]$, we get the 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ \boxed{a_{21} \quad a_{22}} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} b_{11} \\ b_{21} \end{matrix}} & \begin{matrix} b_{12} \\ b_{22} \end{matrix} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ \boxed{a_{21}b_{11} + a_{22}b_{21}} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

Again, I emphasize that the i-th row and j-th column of the product is the dot product of the i-th row of A and the j-th column of B. For example, in the 2×2 case, let us look at the second row and first column. The second row of A is $\begin{bmatrix} a_{21} & a_{22} \end{bmatrix}$ while the first column of B is $\begin{bmatrix} b_{11} & b_{21} \end{bmatrix}^T$. Taking their dot product gives $a_{21}b_{11} + a_{22}b_{21}$ which is indeed the $(2, 1)$ entry of the product.

Example 1.27

Determine the matrix product AB where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & -1 \\ 2 & -3 & -1 \end{bmatrix}$$

Solution. The matrix A has dimension 2×3 while B has dimension 3×3. Their product AB can therefore be computed, and will output a 2×3 matrix. Carrying out the multiplication, we get

$$AB = \begin{bmatrix} 1 + 0 + 4 & 0 + 0 + -6 & 3 + 0 + -2 \\ 3 + 0 + 0 & 0 + 0 + 0 & 9 + 2 + 0 \end{bmatrix} = \begin{bmatrix} 5 & -6 & 1 \\ 3 & 0 & 11 \end{bmatrix}. \qquad \blacksquare$$

A very special type of matrix is the identity matrix. If n is a positive integer, then we define I_n to be the $n \times n$ matrix with 1's on the diagonal and zero everywhere else; that is,

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Theorem 1.28

If A is $m \times n$, B is $n \times k$, and C is $k \times \ell$, then

1. $A(BC) = (AB)C$
2. $I_m A = A I_n = A$
3. $(AB)^T = B^T A^T$

Note the interchange of order than occurs in the transpose; $(AB)^T = B^T A^T$. In fact, this must happen to ensure that the dimensions like up correctly. Since A is an $m \times n$ matrix and B is $n \times k$, their product AB is an $m \times k$ matrix. The transpose is $k \times m$, which comes from multiplying B^T with dimension $k \times n$ against A^T with dimension $n \times m$.

Furthermore, matrix multiplication is distributive:

$$A(B+C) = AB + AC, \qquad (A+B)C = AC + BC$$

where the dimensions of the matrices are chosen so that this makes sense.

Example 1.29

Show that the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix}$$

satisfies the equation $A^2 - A - 10I_2 = 0$.

Solution. Computing A^2 we get

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1+8 & 4-8 \\ 2-4 & 8+4 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -2 & 12 \end{bmatrix}$$

so that

$$A^2 - A - 10I_2 = \begin{bmatrix} 9 & -4 \\ -2 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 9+1-10 & -4+4-0 \\ -2+2-0 & 12-2+10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

as required. ■

Something very nice happens when we multiply a matrix and a column vector. Suppose for example that

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and we wish to compute the prod $A\mathbf{x}$, which yields

$$A\mathbf{x} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

but this is precisely the coefficient set of a linear system, also written as a linear combination of the columns $\mathbf{a}_i$ of A. Hence if $\mathbf{b}$ is the column vector of constants, any linear system is equivalent to solving $A\mathbf{x} = \mathbf{b}$.

Example 1.30

Determine the product $A\mathbf{s}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \qquad \mathbf{s} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Solution. Using matrix multiplication we get

$$A\mathbf{s} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+1+2 \\ 1+2+6 \\ 2+3+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix}.$$

If we let $\mathbf{b} = \begin{bmatrix} 4 & 9 & 7 \end{bmatrix}^T$, then this is precisely the statement that $\mathbf{s}$ is the solution to the linear system given in example 1.6. ■

Theorem 1.31

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ in an $n \times 1$ column vector. The vector $\mathbf{x}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h$ is a solution to the homogeneous linear system $A\mathbf{x} = 0$, and $\mathbf{x}_p$ is a particular solution to $A\mathbf{x} = \mathbf{b}$.

Proof. Let's begin by supposing that $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_p$ is any other solution. Define $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$, in which case

$$A\mathbf{x}_h = A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

showing that $\mathbf{x}_h$ is a solution to the homogeneous system. Rearranging, we thus have $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$ as required.

In the other direction, suppose that $\mathbf{x}_h$ and $\mathbf{x}_p$ satisfy $A\mathbf{x}_h = \mathbf{0}$ and $A\mathbf{x}_p = \mathbf{b}$. Now

$$A(\mathbf{x}_h + \mathbf{x}_p) = A\mathbf{x}_h + A\mathbf{x}_p = \mathbf{0} + A\mathbf{x}_p = \mathbf{b},$$

which is what we wanted to show. □

We've already seen that a homogeneous solution admits basic solutions. To employ Theorem 1.31, we employ a similar strategy, by separating out the column vectors which correspond to the parameters. Everything left over will correspond to the particular solution.

Example 1.32

Solve the homogeneous system

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 1 \\ -2x_1 - 3x_2 + 4x_3 + 5x_4 &= 6 \\ 2x_1 + 4x_2 - 2x_4 &= 2 \end{aligned}$$

and write it as a linear combination of the solution to the homogeneous system and a particular solution.

Solution. Writing this as a matrix and converting to RREF yields

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 1 \\ -2 & -3 & 4 & 5 & 6 \\ 2 & 4 & 0 & -2 & 2 \end{array}\right] \xrightarrow[(-2)R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{(-2)R_2+R_1\to R_1} \left[\begin{array}{rrrr|r} 1 & 0 & -8 & -7 & -15 \\ 0 & 1 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The leading ones occur at x_1 and x_2, so let $x_3 = s$ and $x_4 = t$ be parameter's, so that

$$\begin{aligned} x_4 &= t \\ x_3 &= s \\ x_2 &= 8 - 4x_3 - 3x_4 = 8 - 4s - 3t \\ x_1 &= -15 + 8x_3 + 7x_4 = -15 + 8s + 7t \end{aligned}$$

or written in terms of vectors, by grouping parameters

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -15 \\ 8 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_p} + \underbrace{\begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix} t}_{\mathbf{x}_h}.$$

As indicated, $\mathbf{x}_p$ is a particular solution to the linear system, while $\mathbf{x}_h$ is the general solution to the corresponding homogeneous system. ∎

So matrix multiplication does satisfy many familiar properties of multiplication. However, it also satisfies some very different properties that what we are used to. For example, it is possible for A to be a non-zero matrix, and $\mathbf{v}$ to be a non-zero vector, but still have $A\mathbf{v}$ be the zero vector. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is emblematic of a deeper problem: We can have $A\mathbf{v} = A\mathbf{w}$ but $\mathbf{v} \neq \mathbf{w}$. For example

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Additionally, matrix multiplication is not *commutative*; that is, generally $AB \neq BA$. To see this, let

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

for which

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \\ BA &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

These are not even close to being the same matrix. Finally, powers of non-zero matrices can be zero. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then A is certainly not the 0 matrix, but

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so we have $A^2 = 0$.

1.5 Linear Transformations

Definition 1.33

A *linear transformation* is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

1. $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
2. $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

Namely, linear transformations are functions which preserve addition and scalar multiplication, the two operations we know we can perform on vectors. For example, consider $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}.$$

We can check this is linear as follows: Let $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$. Let's first show that $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, for which

$$\begin{aligned} T(\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right) &= \begin{bmatrix} (x_1 + y_1) + (x_2 + y_2) \\ (x_2 + y_2) + (x_3 + y_3) \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_2 + y_3 \end{bmatrix} = T(\mathbf{x}) + T(\mathbf{y}). \end{aligned}$$

Similarly, if $c \in \mathbb{R}$, then

$$T(c\mathbf{x}) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + cx_2 \\ cx_2 + cx_3 \end{bmatrix} = c\begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} = cT(\mathbf{x}).$$

On the other hand, a function like $T : \mathbb{R}^2 \to \mathbb{R}$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2 + 1$$

is **not** linear. In fact, neither of the two properties required for a linear transformation hold. You should check this on your own.

Proposition 1.34

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.

Proof. There are a couple of ways to proceed. For example, note that

$$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) = 2T(\mathbf{0}).$$

The only vector which could satisfy $\mathbf{x} = 2\mathbf{x}$ is $\mathbf{x} = \mathbf{0}$, showing that $T(\mathbf{0}) = \mathbf{0}$ as required. $\square$

Remember way back in Section 1.1 when I defined a linear map? Matrices are the key to linear maps. For example, suppose that A is an $m \times n$ matrix. This means that it eats $n \times 1$ column vectors, and produces $m \times 1$ column vectors. This is precisely what a function does, and we can define $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\mathbf{x}) = A\mathbf{x}$. This map is linear, since for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} T_A(\mathbf{x} + \mathbf{y}) &= A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} \\ &= T_A(\mathbf{x}) + T_A(\mathbf{y}) \\ T_A(c\mathbf{x}) &= A(c\mathbf{x}) = cA\mathbf{x} \\ &= cT_A(\mathbf{x}). \end{aligned}$$

For example, if A is the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

and $\mathbf{x} = (x_1, x_2, x_3)^T$ then

$$T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 + x_3 \end{bmatrix}.$$

We call maps like T_A *linear transformations*, precisely because you can think of them as transforming a vector $\mathbf{x} \in \mathbb{R}^n$ into a vector in $\mathbf{y} \in \mathbb{R}^n$. When $T_A : \mathbb{R}^n \to \mathbb{R}^n$ – so A is a square matrix – then T_A can be though of as a transformation of $\mathbb{R}^n$ itself. Simple geometric transformations $\mathbb{R}^2 \to \mathbb{R}^2$ include

Scaling: If $a, b \in \mathbb{R}$ then the linear transformation T_A given by

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

scales the x-direction by a and the y-direction by b. If $a < 0$ then T_A also reflects about the y-axis, and when $b < 0$ the transformation reflects about the x-axis.

Rotation: Given an angle θ, we can rotate a vector $\mathbf{x} \in \mathbb{R}^2$ counterclockwise using the T_A where

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Reflection: We can reflect about the line $y = mx$ using the transformation T_A given by

$$A = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Proposition 1.35

If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is $T_A(\mathbf{x}) = A\mathbf{x}$ then for any $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ we have

$$T_A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = c_1 T_A(\mathbf{x}_1) + c_2 T_A(\mathbf{x}_2) + \cdots + c_k T_A(\mathbf{x}_k).$$

So we know that every matrix A defines a linear map $T_A(\mathbf{x}) = A\mathbf{x}$. Importantly, the converse is also true; that is, every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is T_A for an $m \times n$ matrix A. To see that this is the case, and to learn how to find the matrix A, we need the following:

Definition 1.36

If n is a positive integer, we define the *standard basis* for $\mathbb{R}^n$ as the collection $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where

$$\mathbf{e}_i = (\underbrace{0, 0, \ldots, 0, 1}_{i\text{-times}}, 0, \ldots 0).$$

Remark 1.37 The word *basis* here refers to a more general notion of a basis for a vector space, something we will see much later in the course. This is just one of infinitely bases for $\mathbb{R}^n$. I would be remiss if I did not add that what we are about to do is considered by some as an "act of violence." Namely, I am about to tell you how to write down a linear transformation as a matrix using the standard basis, but linear transformations can be written in any basis.

We can write any vector $\mathbf{x} \in \mathbb{R}^n$ as a linear combination of the elements of the standard basis. For example, if $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$ then

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

This is the smallest number of vectors we can use to write down any other vector, and every vector can be written as such, so this makes the collection $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ special.

Now let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\mathbf{e}_i$ be the standard basis for $\mathbb{R}^n$. To write T as a matrix, it suffices to see what T does to the standard basis, for if $\mathbf{x} = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix}^T$ then

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + \cdots + x_n T(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & \ldots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \end{aligned}$$

Hence by setting $A = \begin{bmatrix} T(\mathbf{e}_1) & \ldots & T(\mathbf{e}_n) \end{bmatrix}$ we get that $T(\mathbf{x}) = A\mathbf{x}$ as required.

Example 1.38

Let T be the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x_1, x_2, x_3) = (3x_1 + x_2, x_1 + x_3, x_1 - 4x_3).$$

Show that T is a linear transformation, and find the matrix representation of T.

Solution. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, with $c \in \mathbb{R}$. Checking scalar multiplication is easiest, so we start with that:

$$\begin{aligned} T(c\mathbf{a}) &= T(ca_1, ca_2, ca_3) \\ &= (3ca_1 + ca_2, ca_1 + ca_3, ca_1 - 4ca_3) \\ &= c(3a_1 + a_2, a_2 + a_3, a_1 - 4a_3) \\ &= cT(\mathbf{a}). \end{aligned}$$

Additivity requires a bit more work:

$$\begin{aligned} T(\mathbf{a} + \mathbf{b}) &= T(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= \big(3(a_1 + b_1) + (a_2 + b_2), (a_1 + b_1) + (a_3 + b_3), (a_1 + b_1) - 4(a_3 + b_3)\big) \\ &= (3a_1 + a_2, a_1 + a_3, a_1 - 4a_3) + (3b_1 + b_2, b_1 + b_3, b_1 - 4b_3) \\ &= T(\mathbf{a}) + T(\mathbf{b}). \end{aligned}$$

So T is linear, which means we can write it as $T(\mathbf{x}) = A\mathbf{x}$ for some 3×3 matrix A. Let's see what it does to the standard basis vectors:

$$T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}.$$

Since A is the matrix formed by concatenating these columns, we get

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -4 \end{bmatrix},$$

and we conclude that $T(\mathbf{x}) = A\mathbf{x}$. ■

Remark 1.39 Note that the identity matrix I_n yields the identity function $T_{I_n}(\mathbf{x}) = \mathbf{x}$, and vice-versa. Hence the name.

Matrix multiplication is defined in such a funny way to ensure that compositions of linear transformations make sense. More precisely, let A be an $m \times n$ matrix, and B be an $n \times k$ matrix. Define $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\mathbf{x}) = Ax$ and $T_B : \mathbb{R}^k \to \mathbb{R}^n$ by $T_B(\mathbf{x}) = B\mathbf{x}$. The composition function is

$$T_A \circ T_B : \mathbb{R}^k \to \mathbb{R}^n \to \mathbb{R}^m, \quad (T_A \circ T_B)(\mathbf{x}) = T_A(T_B(\mathbf{x})) = T_A(B\mathbf{x}) = AB\mathbf{x}.$$

Suppose we did not know how to multiply matrices, but wanted to define it so that this makes sense. We need to see what $T_A \circ T_B$ does to the standard basis vectors e_i. Let $\mathbf{b}_i$ be the columns of B, so that

$$(T_A \circ T_B)(\mathbf{e}_i) = AB\mathbf{e}_i = A\mathbf{b}_i,$$

so the matrix AB is

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \ldots & A\mathbf{b}_k \end{bmatrix}.$$

If you think about it, you'll see that this is exactly the definition of matrix multiplication we had above.

1.6 Matrix Inversion

We use inversion to reverse an operation. For example, given the equation $ax = b$ for $a \neq 0$, to solve for x we multiply both sides by a^{-1} to get

$$a^{-1}ax = a^{-1}b \quad \Rightarrow \quad x = a^{-1}b.$$

We would like to do something similar for matrices.

Definition 1.40

Let A be a $n \times n$ matrix. We say that A is *invertible* with inverse B if

$$AB = I_n, \qquad BA = I_n.$$

From the linear transformation point of view, the inverse matrix corresponds to the inverse function. For example, suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by the matrix A, and is invertible with inverse $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ represented by the matrix B. Together, these must satisfy

$$(T \circ T^{-1})(\mathbf{x}) = \mathrm{id}(\mathbf{x}) = \mathbf{x}, \quad (T^{-1} \circ T)(\mathbf{x}) = \mathrm{id}(\mathbf{x}) = \mathbf{x}.$$

Remark 1.39 told us $\mathrm{id}(\mathbf{x}) = I_n\mathbf{x}$, so the above two equations amount to

$$AB = I_n, \qquad BA = I_n$$

which is precisely the definition of the inverse matrix.

We often denote the inverse of A by A^{-1}. This does precisely what we want in terms of solving linear systems: Given an linear system $A\mathbf{x} = \mathbf{b}$ such that A is invertible, we can apply its inverse A^{-1} to both sides to get

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{x} = A^{-1}\mathbf{b}.$$

However, unlike real numbers, not all non-zero matrices have inverses. For example, you can show that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

does not have an inverse by explicitly trying to compute one. In the special case of 2×2 matrices, the inverse is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.4}$$

We can check my multiplying:

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with $A^{-1}A$ similar.

Notice we cannot apply (1.4) to the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

earlier, which I told you was not invertible. We cannot apply the formula precisely because $ad-bc = 0$. It turns out that a 2×2 matrix is invertible if and only if $ad-bc \neq 0$. This generalizes to something known as the *determinant*, which we will discuss later.

Example 1.41

Solve the linear system $A\mathbf{x} = \mathbf{b}$ if

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}.$$

Solution. This is the same linear system as Example 1.4, and there we found the solution $(x_1, x_2) = (1, 3)$. By (1.4) the inverse of A is given by

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

Applying this to $A\mathbf{x} = \mathbf{b}$ to solve for $\mathbf{x}$, we get

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -7 \\ 5 \end{bmatrix} = \begin{bmatrix} -14+15 \\ -7+10 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \qquad \blacksquare$$

Proposition 1.42

If A is an $n \times n$ matrix and B, C are both inverses of A, then $B = C$; that is, inverses are unique.

Proof. By definition, we know both $AB = I_n = BA$ and $AC = I_n = CA$, so

$$B = I_nB = (CA)B = C(AB) = CI_n = C$$

as required. $\square$

There are formulas for inverting 3×3 and higher matrices, but in general they are too messy to be worth remembering. Instead, let $\mathbf{e}_i$ be the standard basis for $\mathbb{R}^n$, and write the columns of

A^{-1} as $\mathbf{f}_i$. The equation $AA^{-1} = I_n$ is equivalent to

$$\begin{aligned} I_n &= \begin{bmatrix} \mathbf{e}_1 & \ldots & \mathbf{e}_n \end{bmatrix} \\ &= AA^{-1} \\ &= A \begin{bmatrix} \mathbf{f}_1 & \cdots & \mathbf{f}_n \end{bmatrix} \\ &= \begin{bmatrix} A\mathbf{f}_1 & \cdots & A\mathbf{f}_n \end{bmatrix}. \end{aligned}$$

By equating, we want to solve the linear system $A\mathbf{f}_i = \mathbf{e}_i$ to find the $\mathbf{f}_i$. We know we can do this with the augmented matrix $\left[A \mid \mathbf{f}_i\right]$, but rather than have to do this for every $\mathbf{f}_i$, we can do them all simultaneously by using the augmented matrix

$$\left[\begin{array}{c|cccc} A & \mathbf{f}_1 & \mathbf{f}_2 & \cdots & \mathbf{f}_n \end{array}\right].$$

If the left portion of the augmented matrix cannot be reduces to the identity matrix, then the matrix is not invertible.

Example 1.43

Find A^{-1} and use it to solve the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix}.$$

Solution. This is the same linear system given in Example 1.6, where we found a solution of $(x_1, x_2, x_3) = (1, 1, 2)$. Setting up our augmented system and row reducing, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-2)R_1+R_3 \to R_3]{(-1)R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 \end{array}\right] \xrightarrow{(-1/3)R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & -1/3 \end{array}\right]$$

$$\xrightarrow[(-1)R_3+R_1 \to R_1]{(-2)R_3+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -5/3 & 1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & -1/3 \end{array}\right] \xrightarrow{(-1)R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/3 & -2/3 & -1/3 \\ 0 & 1 & 0 & -5/3 & 1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & -1/3 \end{array}\right]$$

For simplicity, we factor out the $1/3$ term and write

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 7 & -2 & -1 \\ -5 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

We can then solve the linear system as

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{3} \begin{bmatrix} 7 & -2 & -1 \\ -5 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 28 - 18 - 7 \\ -20 + 9 + 14 \\ 4 + 9 - 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

which agrees with what we found earlier. ■

Example 1.43 was significantly more difficult than Example 1.6 where we just used row reduction. Why then would we ever want to compute the inverse? The problem with row redact is that, were to to change the constants in $\mathbf{b}$, we would have to do the entire row reduction over again. On the other hand, computing the inverse is a one-time thing. Once you have it, you can very quickly solve $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$. So it depends on whether you need to solve $A\mathbf{x} = \mathbf{b}$ for many different $\mathbf{b}$.

Theorem 1.44

Suppose that each A_i is an invertible $n \times n$ matrix.

1. $(A^{-1})^{-1} = A$
2. $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$
3. $(A^k)^{-1} = (A^{-1})^k$ for all k
4. $(cA)^{-1} = (1/c)A^{-1}$ for $c \neq 0$
5. A invertible if and only if A^T invertible.
6. $(A^{-1})^T = (A^T)^{-1}$.

There are a few ways to tell whether a matrix is invertible (and this list will grow with time):

Theorem 1.45

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is invertible,
2. There is a matrix B such that $AB = BA = I_n$,
3. The system $A\mathbf{x} = 0$ has only the trivial solution,
4. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution,
5. The rank of A is n,
6. The reduced row echelon form of A is I_n.

Remark 1.46

1. Computing inverses using Gaussian elimination is actually a very bad way of computing inverses. Modern computers use more sophisticated techniques to compute inverse.
2. Almost every $n \times n$ matrix is invertible. What I mean by this is that if you created an $n \times n$ matrix by randomly choosing the entries, it would be mathematically impossible for you to create a non-invertible matrix. The word 'random' here is important though. Certainly we can construct non-invertible matrices if we are allowed to choose the entries within the matrix.

Example 1.47

Suppose that

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad A^{-1} = \frac{1}{3}\begin{bmatrix} 7 & -2 & -1 \\ -5 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

If $B = ADA^{-1}$, compute B^{-1} and $B^2 + B$.

Solution. Using brute force, you could explicitly compute B, then apply our algorithm above for computing the inverse, but this is a lot of work. Using our properties of inversion, we can simplify the process. For example,

$$B^{-1} = (ADA^{-1})^{-1} = (A^{-1})^{-1}D^{-1}A^{-1} = AD^{-1}A^{-1}.$$

Since D is a diagonal matrix, its inverse is just the reciprocal of the diagonal entries, so

$$\begin{aligned} B^{-1} = AD^{-1}A^{-1} &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 7 & -2 & -1 \\ -5 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1 & 1 & 1 \\ 2 & 3/2 & 1/3 \end{bmatrix}\begin{bmatrix} 7 & -2 & -1 \\ -5 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 6 & 3 & 2 \\ 6 & 6 & 6 \\ 12 & 9 & 2 \end{bmatrix}\begin{bmatrix} 7 & -2 & -1 \\ -5 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \\ &= \frac{1}{18}\begin{bmatrix} 29 & -7 & -2 \\ 18 & 0 & 0 \\ 41 & -13 & 4 \end{bmatrix}. \end{aligned}$$

Similarly, note that

$$B^2 = (ADA^{-1})^2 = (ADA^{-1})(ADA^{-1}) = AD^2A^{-1}$$

with D^2 computed easily as the square of the elements on the diagonal. Thus

$$B^2 + B = AD^2A^{-1} + ADA^{-1} = A(D^2 + D)A^{-1},$$

which can be computed as

$$B^2 + B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}\begin{bmatrix} 7 & -2 & -1 \\ -5 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -4 & 14 & -2 \\ -10 & 44 & -14 \\ -50 & 22 & 20 \end{bmatrix}.$$ ■

2 Complex Numbers

For various reasons which I will enumerate below, it is useful to introduce a new number which squares to negative one. We define the *imaginary number i* formally as the number which satisfies $i^2 = -1$. This is called imaginary because no real number satisfies this property. It is worth noting that this number shows up very often in real life applications, and should not be construed as something make believe construct of mathematicians. Note that this allows us to take the square root of any negative number, since if $c \in \mathbb{R}$ then $(ci)^2 = c^2 i^2 = -c^2$.

The *complex numbers*, denoted $\mathbb{C}$, are the collection of all numbers of the form $z = a + bi$ for $a, b \in \mathbb{R}$. Here we say that a is the *real part* of z, denoted $\Re(z)$, while b is the imaginary part, denoted $\Im(z)$. These can be visualized in the complex plane, by identifying $a + bi$ with (a, b).

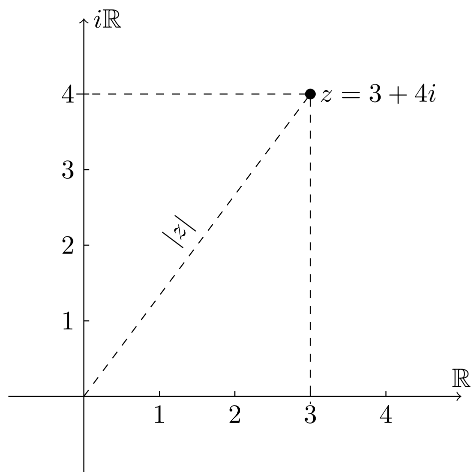

Figure 3: Visualizing $\mathbb{C}$ as a plane. Every element $a + bi$ is identified with (a, b), so that the x-axis is known as the *real* axis, and the y-axis is known as the *imaginary* axis.

2.1 Properties of the Complex Numbers

The addition of two complex numbers is done by combining the real and imaginary components separately. For example, if $z = a + bi$ and $w = c + di$ then

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

It's not too hard to see that $z + w = w + z$ and $z + 0 = z$.

We define the product of two complex numbers by demanding that multiplication be distributive across addition, just like in the real numbers. For example, if $z = a + bi$ and $w = c + di$ then

$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i, \end{aligned}$$

which is again of the form (real) + (real)i. In fact, since $i^2 = -1$, any complex number can be reduced to the above form. You can check that multiplication is commutative; that is, $zw = wz$.

Example 2.1

Let $u = 3 - 2i$ and $v = -7 + 3i$. Compute

1. $u - iv$,
2. uv,
3. $u^2 - v^2$.

Solution. In every case, we write everything as $a + bi$ to take advantage

1. Note that $iv = i(-7 + 3i) = -7i + 3i^2 = -3 - 7i$, so
$$u - iv = (3 - 2i) - (-3 - 7i) = 6 + 5i.$$
2. Applying our formula from above
$$uv = (3 - 2i)(-7 + 3i) = [(3)(-7) - (-2)(3)] + [(-2)(-7) + (3)(3)]\, i = -15 - 5i.$$
3. We begin by computing each of u^2 and v^2 separately:
$$\begin{aligned} u^2 &= (3 - 2i)(3 - 2i) = 5 - 12i \\ v^2 &= (-7 + 3i)(-7 + 3i) = 40 - 42i. \end{aligned}$$
Now these can be added to yield
$$u^2 - v^2 = (5 - 12i) - (40 - 42i) = -35 + 30i. \qquad \blacksquare$$

If $z = a + bi$ we define the complex conjugate $\bar{z} = a - bi$. The conjugate plays together nicely with z, since
$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2,$$
which is always a real number. From this we define the *modulus of* z, as $|z| = \sqrt{z\bar{z}}$. The geometric interpretation of the modulus is the length of the vector (a, b). This can be seen in Figure 3.

Example 2.2

If $z = 3 + 4i$, determine $|z|$.

Solution. Multiplying by the conjugate $\bar{z} = 3 - 4i$ or applying the formula for $z\bar{z}$, we get
$$z\bar{z} = 3^2 + 4^2 = 25.$$
The modulus is the square root of this, so $|z| = 5$. $\blacksquare$

Remark 2.3 You must be very careful when applying familiar operations on complex numbers. For example, we defined the modulus of z to be $|z| = \sqrt{z\bar{z}}$; however, this is *not* the same thing as $\sqrt{z}\sqrt{\bar{z}}$. The fact that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ **only holds if** $a, b \in \mathbb{R}$. If we assumed that this identity did hold, then

$$(-1)(-1) = 1 \quad \Rightarrow \quad \sqrt{(-1)(-1)} = 1 \quad \overset{\text{Wrong}}{\Rightarrow} \quad \sqrt{-1}\sqrt{-1} = 1 \quad \Rightarrow \quad i^2 = 1$$

But $i^2 = -1$, so this cannot be true.

The fact that $i^2 = -1$ means that $1/i = -i$. More generally, we can use the conjugate to help us determine $1/(a+bi)$. If $z = a + bi$ then

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} \quad \Rightarrow \quad \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i,$$

which is in standard form.

Example 2.4

If $z = 1 - 2i$ and $w = 3 + i$, determine w/z and write it in standard form.

Solution. To compute $1/z$ we apply the above formula or recompute from scratch:

$$\frac{1}{z} = \frac{1}{1-2i} = \frac{1+2i}{5} = \frac{1}{5} + \frac{2}{5}i.$$

Multiplying by w gives

$$\frac{w}{z} = \frac{1}{5}(3+i)(1+2i) = \frac{1}{5}\left[(3-2) + (6+1)i\right] = \frac{1}{5} + \frac{7}{5}i. \qquad \blacksquare$$

Proposition 2.5

If $z, w \in \mathbb{C}$, then

1. $\overline{z \pm w} = \bar{z} \pm \bar{w}$,
2. $\overline{zw} = \bar{z}\,\bar{w}$,
3. $\overline{z^{-1}} = \bar{z}^{-1}$,
4. $\bar{\bar{z}} = z$
5. $|zw| = |z||w|$
6. $|z^{-1}| = |z|^{-1}$.
7. $|z + w| \leq |z| + |w|$

Proof. I'll give the proof for (2), and leave the rest as an exercise for you to check. Let $z = a + bi$ and $w = c + di$. The left hand side of $\overline{zw} = \overline{zw}$ gives

$$\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd) + (ad+bc)i} = (ac-bd) - (ad+bc)i.$$

The right hand side gives

$$\bar{z}\,\bar{w} = (\overline{a+bi})(\overline{c+di}) = (a-bi)(c-di) = (ac-bd) - (ad+bc)i.$$

Both sides are equal, so $\overline{zw} = \overline{zw}$ as required. $\square$

As a bonus, there is a way of identifying the complex numbers with matrices in such a way that matrix multiplication is the same as complex multiplication. If $z = a + bi$ then let Z be the matrix

$$Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Addition of matrices is done component wise, so certainly agrees with complex addition. For example, if $w = c + di$ then

$$Z + W = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a + c & -(b + d) \\ b + d & a + c \end{bmatrix}$$

which agrees with the matrix for $z + w = (a + c) + (b + d)i$. The more interesting fact is that this also preserves complex multiplication:

$$ZW = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}$$

which corresponds to the matrix for $zw = (ac - bd) + (ad + bc)i$. This is one of the ways in which computers do computations with complex numbers.

2.2 Some Motivation

Why are complex numbers useful? As mentioned above, they appear naturally in many fields of study, but let's motivative them using polynomials. We know that some polynomials do not have roots. For example, $x^2 + 1 \neq 0$ for any $x \in \mathbb{R}$. More generally, the polynomial

$$ax^2 + bx + c$$

has a real root if and only if $b^2 - 4ac \geq 0$, which follows from the quadratic formula. Viewed another way, every polynomial with real coefficients can be factored into at worst linear and quadratic factors. We have to allow quadratics precisely because of terms like $x^2 + 1$ which cannot be factored into linear terms.

However, when we allow ourselves to work over $\mathbb{C}$ instead of $\mathbb{R}$, every quadratic polynomial has a root, and every polynomial can be factored into a product of linear terms. Indeed, by the quadratic formula, the roots of $ax^2 + bx + c$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 - 4ac \geq 0$ we get our normal roots, but when $b^2 - 4ac < 0$ we are taking the square root of a negative number, which is resolved using complex numbers:

$$ax^2 + bx + c = \left(x - \underbrace{\frac{-b + i\sqrt{4ac - b^2}}{2a}}_{\in\mathbb{C}} \right) \left(x - \underbrace{\frac{-b - i\sqrt{4ac - b^2}}{2a}}_{\in\mathbb{C}} \right).$$

Theorem 2.6: The Fundamental Theorem of Algebra

If p is any polynomial with coefficients in $\mathbb{C}$, then p has a root. Inductively, p can thus be written as a product of linear factors.

When our polynomial has real coefficients, something else can be said.

Theorem 2.7

Suppose that p is a polynomial with real coefficients. If z is a root of p, then $\bar{z}$ is a root of p.

Proof. Suppose that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. By assumption, $p(z) = 0$; that is,

$$0 = p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Taking complex conjugates gives

$$\begin{aligned} 0 = \overline{p(z)} &= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \\ &= \overline{a_n}\,\overline{z}^n + \overline{a_{n-1}}\,\overline{z}^{n-1} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} \\ &= a_n \overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_1 \overline{z} + a_0 && \text{since } a_i \in \mathbb{R} \\ &= p(\bar{z}). \end{aligned}$$

Since $p(\bar{z}) = 0$, this shows that $\bar{z}$ is also a root of p. ☐

The fact that every polynomial over $\mathbb{C}$ can be factor into linear terms has a special property. We say that $\mathbb{C}$ is an *algebraically closed field*. Algebraically closed fields are exceptionally important in mathematics.

2.3 The Polar Form

There is a second way of parameterizing complex numbers which is often far more useful than the standard form above. We saw that $z = a + bi$ could be identified with the point $(a, b) \in \mathbb{R}^2$ by thinking of (a, b) in Cartesian coordinates. However, there is another coordinate system on $\mathbb{R}^2$, called *polar coordinates*, which assigns to a point the value (r, θ), where r is the distance from the origin to the point, and θ is the angle subtended by the line through the origin and the positive x axis. This is shown in Figure 4.

The conversion between Cartesian and polar coordinates is as follows. If (x, y) is a point in Cartesian coordinates, then

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan(y/x).$$

If (r, θ) is a point in polar coordinates, then

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

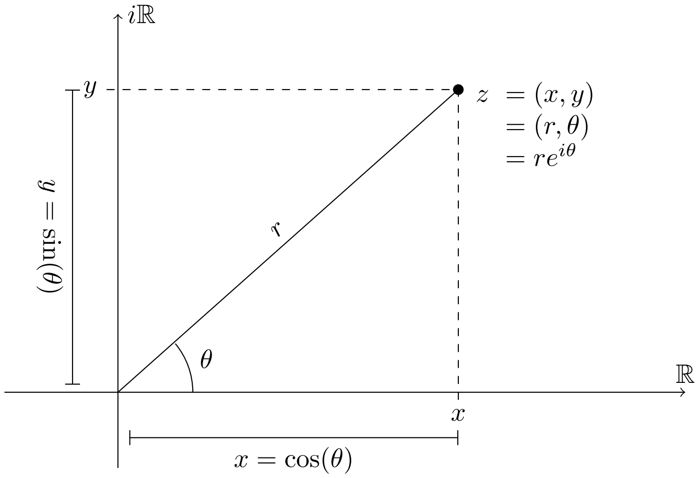

Figure 4: A point the complex plane can be thought of in polar coordinates rather than Cartesian coordinates. In that case, the relation $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ can be used to express the coordinate (r, θ) as $re^{i\theta}$.

There is a nice relationship between the two. If you have not yet seen the topic of Taylor series, ignore this next part and just read Equation (2.1). The Taylor series of the exponential function e^x about 0 is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

with interval of convergence $\mathbb{R}$. There is some hand waving as why we can plug in complex numbers, but suppose we substitute $x = i\theta$. Notice that

$$i^0 = 1, \qquad i^1 = i, \qquad i^2 = -1, \qquad i^3 = -i,$$

after which the cycle repeats. This means that even powers of i are real and odd powers are complex, so

$$\begin{aligned} e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} &= \underbrace{\sum_{m=0}^{\infty} \frac{(i\theta)^{2m}}{(2m)!}}_{\text{even powers}} + \underbrace{\sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!}}_{\text{odd powers}} \\ &= \left[\sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m}}{(2m)!}\right] + i\left[\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}\right] \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

Hence for any $\theta \in \mathbb{R}$, we have

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{2.1}$$

This is an extraordinary identity: Who would ever suspect that the exponential and trigonometric functions would be related? For our purposes though, suppose $z \in \mathbb{C}$ with Cartesian coordinates $z = x + iy$ and polar coordinates (r, θ), so that

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

Writing $z = re^{i\theta}$ is the polar representation of a complex number.

Example 2.8

Write $z = 2 + 2i$ in polar form.

Solution. Here the distance from the origin 0 to z is the modulus of z, which we compute to be

$$|z| = \sqrt{z\bar{z}} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

The angle subtended by this line with the positive real axis is

$$\theta = \arctan(2/2) = \arctan(1) = \frac{\pi}{4}.$$

Hence the polar representation of z is $z = 2\sqrt{2}e^{i\pi/4}$. ■

The great thing about polar form is that the rules for the exponent still hold.

Theorem 2.9

If $z = re^{i\theta}$ and $w = se^{i\psi}$ are complex numbers, then $zw = rse^{i(\theta+\psi)}$.

Corollary 2.10: De Moivre's Theorem

If $n \in \mathbb{Z}$ and $z = re^{i\theta}$ then $z^n = r^n e^{in\theta}$.

Example 2.11

Suppose that $z = 2 + 2\sqrt{3}i$. Determine z^{12}.

Solution. This would be terrible using the Cartesian representation of z, so we first convert this to polar coordinates. Note that

$$r = |z| = \sqrt{4 + 12} = \sqrt{16} = 4,$$

while

$$\theta = \arctan\left(\frac{2\sqrt{3}}{2}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

Thus we can write z as $z = 4e^{i\pi/3}$. Taking the 12th power thus gives

$$z^{12} = r^{12}e^{12i\theta} = 4^{12}e^{12\pi/3} = 4^{12}e^{4\pi} = 4^{12}\left(\cos(4\pi) + i\sin(4\pi)\right) = 4^{12}.$$ ■

We can reverse De Moivre's Theorem to find nth roots as well.

Example 2.12

Find all solutions to $z^5 = 1$ over $\mathbb{C}$.

Solution. Over the real numbers, there is only one solution, $z = 1$. Over the complex numbers however, there are five solutions. If $w = re^{i\theta}$ is a solution to $z^5 = 1$ then

$$1 = 1e^0 = w^5 = r^5 e^{5i\theta}.$$

Immediately, we must have that $r^5 = 1$ so $r = 1$. On the other hand, we want to say that $5\theta = 2\pi$, so that $\theta = \pi/5$, but this would only give us a single solution, and we know there should be 5. The key lies in the fact that if $5\theta = 4\pi$ we get the same result, or similarly if $5\theta = 6\pi$. If we're trying to solve $k\theta = 2n\pi$, then we'll get unique values for θ until $k = n$, in which case we'll start the cycle over again. In this case, we get

$$\theta = 0, \quad \frac{2\pi}{5}, \quad \frac{4\pi}{5}, \quad \frac{6\pi}{5}, \quad \frac{8\pi}{5}.$$

Note that the next value, $10\pi/5 = 2\pi$, is the same answer as $\theta = 0$. After this point, all answers begin to repeat themselves and so are rejected. Thus our solutions are

$$w_0 = 1, \quad w_1 = e^{2\pi i/5}, \quad w_2 = e^{4\pi i/5}, \quad w_3 = e^{6\pi i/5}, \quad w_4 = e^{8\pi 8/5}$$

shown in Figure 5. ■

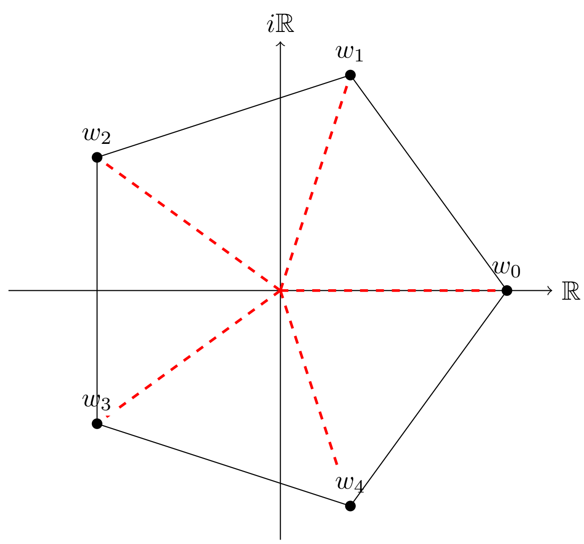

Figure 5: The 5th roots of unity. Notice how they form a regular pentagon.

In general, the solutions to $z^n = 1$ are called the *nth roots of unity*, and are of the form

$$\zeta_{k,n} = e^{2\pi ik/n}, \qquad k = 0, 1, \ldots, n.$$

When plotted in the complex plane, they will always form a regular n-gon. We can use the nth roots of unity to find the solutions to $z^n = a$ for any positive real number a. In that case, the solutions will all be of the form $\sqrt[n]{a}\zeta_{k,n}$ for $k = 0, 1, \ldots, n$.

Example 2.13

Find all complex roots to the equation $z^6 = -64$.

Solution. Suppose $z = re^{i\theta}$ for r and θ yet to be determined. We can write $-64 = 64e^{i\pi}$, so that using de Moivre's theorem we get $r^6 e^{6i\theta} = 64e^{i\pi}$. We immediately know that $r^6 = 64$ implies that $r = 2$, leaving only the values of θ. As with Example 2.12, we know that $6\theta = \pi + 2n\pi = (2n+1)\pi$, which gives use the values

$$\theta = \frac{\pi}{6}, \quad \frac{3\pi}{6}, \quad \frac{5\pi}{6}, \quad \frac{7\pi}{6}, \quad \frac{9\pi}{6}, \quad \frac{11\pi}{6}.$$

The next value $13\pi/6$ gives the same answer as $\pi/6$, so we reject it and all further answers. Thus our solutions are

$$z = 2e^{i(2k+1)\pi/6}, \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5.$$

■

3 Determinants

In this section we analyze the *determinant of a matrix*. Very loosely, the determinant is map which assigns to each matrix a real-number. The value of this real number has several interpretations. Sometimes we care about the magnitude of this number, sometimes the sign, and sometimes we are only interested in whether the number is non-zero. For example, the determinant will give us a way of determining whether a matrix is invertible, without having to explicitly compute the inverse.

Unfortunately, most of the ways of writing down the determinant are quite complicated. The definitions which are theoretically useful are poor for computation, and the definitions which are useful for computation are poor theoretically. Even those which are computationally valuable turn out to be egregiously intensive.

3.1 Definition

As mentioned above, the determinant map which assigns to each matrix a real number. The definition we will use for the determinant will be by *cofactor expansion*, alternatively known as the *Laplace extension*. To begin with, if $A = [a]$ is a 1×1 matrix then its determinant is $\det(A) = a$. If A is a 2×2 matrix, its determinant is defined to be

$$\det(A) = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

the product of the diagonal minus the product of the anti-diagonal. The 3×3 case is far trickier. I will write it down, then comment on precisely how it is calculated. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

for which

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \tag{3.1}$$

This looks esoteric and arbitrary, but there is a method to the madness.

Definition 3.1

Let A be an $n \times n$ matrix. For any $1 \leq i, j \leq n$, the (i,j)-*submatrix* of A, denoted M_{ij}, is the $(n-1) \times (n-1)$ matrix formed by deleting the i-th row and j-th column from A. The (i,j)-cofactor of A, denoted C_{ij}, is $C_{ij} = (-1)^{i+j} \det(M_{ij})$.

Example 3.2

Determine the $(1,3)$- and $(2,3)$-cofactor of $A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution. The $(1,3)$-cofactor is $C_{13} = (-1)^{1+3} \det(M_{13}) = \det(M_{13})$ where M_{13} is the submatrix formed by deleting the first row and third column of A, hence

$$C_{13} = \det \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} = (3 \times 1) - (-1 \times 0) = 3.$$

Similarly, the $(2,3)$-cofactor is $C_{23} = (-1)^{2+3} \det(M_{23}) = -\det(M_{23})$, so

$$C_{23} = -\det \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = -\left[(1 \times 1) - (4 \times 0)\right] = -1.$$ ■

Notice that we can write (3.1) as

$$\begin{aligned} \det(A) &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{32}\right) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{21}(-1) \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}. \end{aligned}$$

That is, the determinant of the 3×3-matrix was a weighted sum of the cofactors along the first row of the matrix! We certainly do not expect the student to have guessed that this was the case. Instead, this is what we will actually use for the definition. An important point however, is that there is nothing special about the first row. We could use any other row or column. For example, the student can check that

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{23}C_{23}$$

yields exactly the same formula as (3.1), where now we have done a weighted sum of cofactors along the first column.

Definition 3.3

If A is an $n \times n$ matrix, then the *determinant* of A is the weighted sum of the cofactors along any row or column. For example, along the i-th row or j-th column:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

There is absolutely no reason that the student should believe that this quantity is invariant regardless of the choice of row or column. The proof of this fact is horrific using the definition that we have given, but the nice proof requires a more abstract and technical definition – an example of what I call 'conservation of mathematical work.'

Example 3.4

Compute $\det \begin{bmatrix} 1 & 4 & -2 \\ 3 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution. Since we have already computed the cofactors C_{13} and C_{23}, it makes most sense to perform our cofactor expansion along the third column. To do this we need to determine C_{33}, which computation yields

$$C_{33} = (-1)^{3+3} \det(M_{33}) = \det \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} = -1 - 12 = -13.$$

Putting this all together, we get

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (-2 \cdot 3) + (0 \cdot -1) + (1 \times -13) = -19. \qquad \blacksquare$$

Notice how the presence of a zero in the $(2,3)$-position made our lives easier? As a general rule, if computing the derivative via cofactor expansion, it makes the most sense to expand along the row/column which contains the most zeroes. In fact, if a matrix has a row or column consisting entirely of zeroes, cofactor expansion along that row/column will always yield a determinant of 0.

Example 3.5

Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 2 & 2 & 2 \end{bmatrix}$.

Solution. Expanding over the first column (since it has the most zeroes), we get

$$\begin{aligned} \det(A) &= (-1)^{1+1}(1) \det \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} + (-1)^{1+2}(0) \det \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} + (-1)^{1+3}(2) \det \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \\ &= [(-2) - (-2)] + 2\,[(-4) - (-3)] \\ &= -2. \end{aligned}$$

Let's see that this answer is actually the same as if we expanded across the second row instead. Here we would get

$$\begin{aligned} \det(A) &= (-1)^{2+1}(0) \det \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} + (-1)^{2+2} \det \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} + (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \\ &= [(2) - (6)] - [(2) - (4)] \\ &= -2. \end{aligned} \qquad \blacksquare$$

Exercise: Compute the determinant of the matrix given in Example 3.5 by expanding along any other row or column, and check to make sure that you got the same answer as computed above.

The presence of many zeroes can make computing the determinant exceptionally simple:

Definition 3.6

A matrix $A = (a_{ij})$ is said to be *upper triangular* if $a_{ij} = 0$ whenever $i > j$.

Stare at this definition for a second to make sense of what it means. A matrix is upper triangular when all of its non-zero entries occur on or above the diagonal. For example, the matrix

$$\begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is upper triangular.

Theorem 3.7

If $A = (a_{ij})$ is upper triangular, then $\det(A) = a_{11}a_{22}\cdots a_{33}$; the product of the diagonal elements.

Proof. Let's proceed by induction on the dimension of the matrix. In the 2×2 case we have

$$\det \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = a_{11}a_{22}$$

as required. Now assume that for an $(n-1) \times (n-1)$ upper triangular matrix, the determinant is the product of the diagonal entries. If A is an $n \times n$ upper triangular matrix, it looks like

$$A = \begin{bmatrix} a_{11} & * \\ 0 & M_{11} \end{bmatrix}$$

where M_{11} is the $(1,1)$-submatrix of A, which has dimension $(n-1) \times (n-1)$. Take the determinant along the first column, we see that the only non-zero term will come from a_{11}, so $\det(A) = a_{11} \det(M_{11})$. By the induction hypothesis, $\det(M_{11})$ is the product of its diagonal components, so

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

as required. □

So for the example given above, the determinant can be read off as

$$\det \begin{bmatrix} 1 & -1 & 4 & 5 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (1)(3)(-5)(1) = -15.$$

Corollary 3.8

If I_n is the $n \times n$ identity matrix, then $\det(I_n) = 1$.

3.2 Properties of the Determinant

Dealing with determinants can be a big pain, so we would like to develop some tools to make our lives a little bit easier. The most useful tool will be the following:

Theorem 3.9

If A, B are two $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Notice the curious fact that the determinant does not care about the order of multiplication, since the product on the right-hand side $\det(A)\det(B) = \det(B)\det(A)$ is an operation in $\mathbb{R}$. We omit the proof of this theorem, but let us compute a few examples to check its veracity.

Example 3.10

Let $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 5 & -2 \end{bmatrix}$. Determine $\det(A)$, $\det(B)$, $\det(AB)$, and $\det(BA)$.

Solution. Straightforward computation yields

$$\det(A) = -7, \quad \det(B) = -3.$$

The product matrices are

$$AB = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 24 & -9 \\ 13 & -4 \end{bmatrix}, \qquad \det(AB) = (-96 + 117) = 21 = \det(A)\det(B),$$

$$BA = \begin{bmatrix} -1 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 19 \end{bmatrix}, \qquad \det(BA) = 19 + 2 = 21 = \det(A)\det(B). \qquad \blacksquare$$

Note however that the determinant is **not** additive; that is, $\det(A + B) \neq \det(A) + \det(B)$. Indeed, almost any pair of matrices will break this. A simple example is to take $A = I_2$ and $B = -I_2$. Then $A+B$ is the zero matrix, so $\det(A+B) = 0$. On the other hand, $\det(A)+\det(B) = 1+1 = 2$.

Exercise: Show that $\det(AB) = \det(A)\det(B)$ explicitly in the 2×2 case.

Corollary 3.11

If A is an invertible $n \times n$ matrix, then $\det(A^{-1}) = 1/\det(A)$.

Proof. We know that $AA^{-1} = I_n$, so applying the determinant we have $\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(I_n) = 1$. Isolating for $\det(A^{-1})$ we get

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

as required. □

Example 3.12

Let A be an invertible matrix such that $A^3 = A$. Show that $\det(A)^2 = \det(A)$.

Solution. Knowing that $A^3 = A$ we can apply the determinant to find $\det(A) = \det(A^3) = \det(A)^3$. Subtracting $\det(A)$ from both sides we get

$$0 = \det(A)^3 - \det(A) = \det(A)\left[\det(A) + 1)\right]\left[\det(A) - 1\right].$$

This can only be true if $\det(A) = \pm 1$ or $\det(A) = 0$, and in either case $\det(A)^2 = \det(A)$. ■

Proposition 3.13

If A is an $n \times n$ matrix, then $\det(A) = \det(A^T)$

The proof of this follows by induction, but we have already done an induction proof today, and that is more than enough!

3.2.1 Quick Aside: Elementary Row Matrices

Recall that there are three operations that we can perform on rows, without changing the row space:

1. We can multiply a row by a non-zero constant,
2. We can interchange two rows,
3. We can add a multiple of one row to another row.

Each of these operations can be represented by left-multiplication with a matrix. To illustrate what these matrices are, consider the 2×2 case. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Multiplying the second row by the number r corresponds to left-multiplication by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$, since

$$\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ rc & rd \end{bmatrix}.$$

Interchanging two rows corresponds to multiplying by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

Finally, adding r times the first row to the second row is given by $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$ since

$$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ra + c & rb + d \end{bmatrix}.$$

We can use these to determine how a matrix transforms under elementary row operations:

Theorem 3.14

If A is an $n \times n$ matrix, then

1. If we scale a row by $r \neq 0$, the corresponding matrix has determinant $r \det(A)$,
2. If we interchange any two rows, the corresponding matrix has determinant $-\det(A)$,
3. If we add a multiple of one row to another, the corresponding matrix has determinant $\det(A)$.

Proof. In the 2×2 case, we have

$$\det \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} = r, \quad \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1, \quad \det \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = 1.$$

Using the fact that $\det(AB) = \det(A)\det(B)$, the result follows. For higher dimensions, these determinant relations are still the same, concluding the proof. □

One may have noticed how laborious it is to compute the determinants of general matrices. Theorem 3.14 tells us that we can reduce the amount of work by using elementary row operations to first reduce the matrix into row echelon form (which is upper triangular), then apply the determinant to the resulting upper triangular matrix. This makes for much less work!

Example 3.15

Compute $\det \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$.

Solution. Reducing to row echelon form, we have

$$\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \xrightarrow[3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & -2 & -4 & -10 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -4 & -10 \end{bmatrix}$$

Since we have a row of zeros, we can immediately conclude that the determinant is 0. ■

This hints at something: We know that if a matrix is not invertible, we will eventually get a row/column with all zeroes in it. This would suggest that we can use the determinant to say something about invertibility.

Corollary 3.16

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof. A matrix A is invertible if and only if, using only row operations, A can be manipulated to reduced row echelon form, say $\tilde{A}$ with 1's on the main diagonal. We know $\det(\tilde{A}) = 1$, and in the process of changing $\tilde{A}$ back to A we can only change signs and multiply by non-zero constants, showing that $\det(A)$ cannot be zero. □

3.3 Determinants and Volume

Determinants have a plethora of applications, the majority of which are beyond the scope of this course. The next section (eigenvalues) presents an important application and we will dedicate a great deal of time to its study. For now, we take a look at an important property of determinants that will manifest in the study of multivariate calculus.

The idea is as follows: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ effectively acts by scaling and rotating vectors. If this is the case, how do volumes transform under T?

For example, let $0 \leq \theta < 2\pi$ be an angle, and consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\mathbf{x} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This rotates a vector by an angle of θ with respect to the x-axis. We do not expect rotation to affect volume. Notice that $\det T = \cos^2(\theta) + \sin^2(\theta) = 1$.

For example, let $a, b \neq 0$, and consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\mathbf{x} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

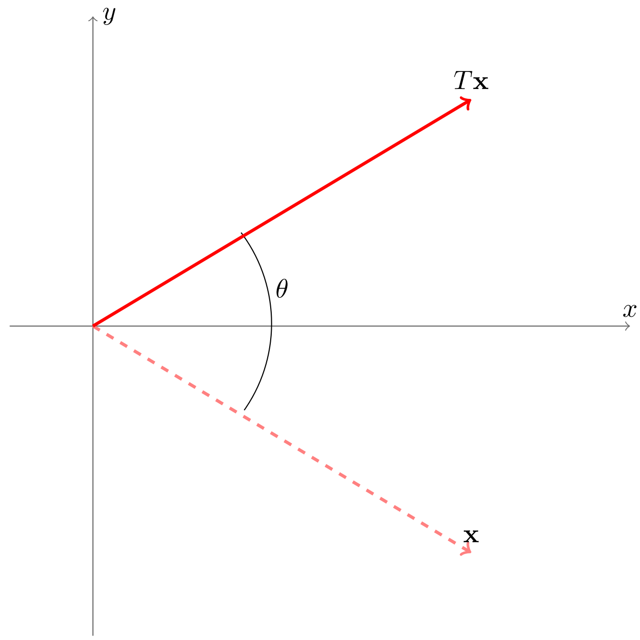

Figure 6: The transformation for a rotation of an angle θ.

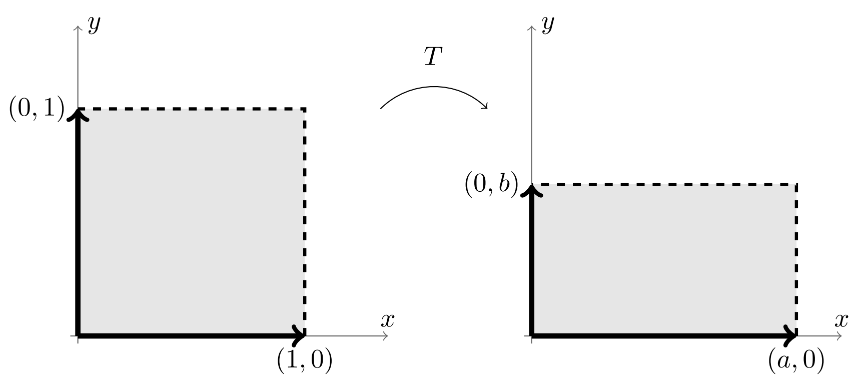

Figure 7: How the unit square transforms under a scaling transformation.

This does not rotate, but instead scales vectors. For example, look at the unit square formed from the vectors $(1, 0)$ and $(0, 1)$, which has area 1. Under this transformation, those vectors become $(a, 0)$ and $(0, b)$, and the resulting square has area ab. Notice that $\det T = ab$.

Similarly, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

then one of the vectors 'collapses.' For example, squares get mapped to lines. Generally, any two dimensional object with area will be mapped to a one dimensional object without area. Notice that $\det T = 0$.

In each of the examples above, the determinant was precisely the amount by which the area of each object scaled. In fact, the following theorem is true:

Theorem 3.17

If $C \subseteq \mathbb{R}^n$ has finite volume and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then the volume of the transformed shape $T(C)$ satisfies

$$\operatorname{Vol}(T(C)) = |\det T| \operatorname{Vol}(C).$$

The proof of this theorem is rather technical and not very enlightening, even in the most simple of cases, hence it is omitted.

Example 3.18

Find the volume of the ellipsoid, centered at the origin, whose x-intercept occurs at $\pm a$, y-intercept occurs at $\pm b$, and z-intercept occurs at $\pm c$.

Solution. The trick here is recognizing how to form the ellipse as the image of a linear transformation. In essence, an ellipse is just a slightly deformed sphere. Since the sphere intercepts each of the x-,y-, and z-axes at ± 1, the transformation that maps the sphere to the ellipsoid is just

$$T = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

We have $\det T = abc$, and the volume of the sphere is $\frac{4}{3}\pi$, so the volume of the ellipsoid $\frac{4}{3}\pi|\det T| = \frac{4}{3}\pi|abc|$. ■

Aside: The relation to calculus comes through 'substitution.' If $y = f(x)$ then when we make substitutions we write $\mathrm{d}y = f'(x)\mathrm{d}x$. In a sense, the $\mathrm{d}x$ and $\mathrm{d}y$ components represent vectors, and the $f'(x)$ is telling us how the length of the vectors change. When dealing with multiple variables, we are often interested in looking at things like $\mathrm{d}u\mathrm{d}v$, where $\mathrm{d}u\mathrm{d}v$ is now the area of a small square. If $(u, v) = F(x, y)$, then the relationship between the squares $\mathrm{d}u\mathrm{d}v$ and $\mathrm{d}x\mathrm{d}y$ is given by $\mathrm{d}u\mathrm{d}v = |\det \mathrm{d}F|\mathrm{d}x\mathrm{d}y$.

3.4 Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are one of the most important applications of linear algebra, since there is a sense in which a matrix is effectively determined by these values. The word 'eigen' comes from the German word 'own,' as in, belong to. The majority of the world uses the word eigenvalues, but it is worth noting that the French write 'values propre,' with propre again being the French word for 'belong to.'

Definition 3.19

Let A be an $n \times n$ matrix. A (real) *eigenvalue* of A is a $\lambda \in \mathbb{R}$ such that there exists a non-zero vector $\mathbf{v}_\lambda$ satisfying

$$A\mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda.$$

In such an instance, we say that $\mathbf{v}_\lambda$ is an *eigenvector* of A corresponding to the eigenvalue λ.

For example, one can check that

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so $\lambda = -2$ is an eigenvalue of this matrix, with associated eigenvector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Notice if we substitute $\begin{bmatrix} 2 & 2 \end{bmatrix}^T$ we would get

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

so that $(2, 2)$ is also an eigenvector, with the same eigenvalue. Interesting! We will see why this is the case in Proposition 3.23 below.

So how do we find eigenvalues and eigenvectors? Recognize that we can re-write $A\mathbf{v}_\lambda = \lambda\mathbf{v}_\lambda$ as $(A - \lambda I)\mathbf{v}_\lambda = 0$. In particular, we are asking that the matrix $(A - \lambda I)$ send a non-zero vector $\mathbf{v}_\lambda$ to the zero vector. This can only happen if $(A - \lambda I)$ is *not* invertible; that is, if $\det(A - \lambda I) = 0$. Computing $c_A(\lambda) = \det(A - \lambda I)$ will result in a polynomial in the variable λ, known as the *characteristic polynomial*. If we can find the roots of this polynomial, we will have the eigenvalues. Moreover, once we know λ, we know that $\mathbf{v}_\lambda$ is a non-trivial solution to $(A - \lambda I)\mathbf{v}_\lambda = \mathbf{0}$. In effect, we've proven the following:

Theorem 3.20

Suppose A is an $n \times n$ matrix.

1. The eigenvalues of λ are precisely the roots of the characteristic polynomial $c_A(\lambda) = \det(A - \lambda I)$
2. Given a value of λ, its corresponding eigenvectors are the non-trivial basic solutions to the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

So let's compute the determinant of $A - \lambda I$ and see what we get. If $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$ as above then

$$\begin{aligned}0 = \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix} \\ &= (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 6 \\ &= (\lambda - 3)(\lambda + 2).\end{aligned}$$

which is zero when $\lambda = 3$ and $\lambda = -2$. We already knew that $\lambda = -2$ via the example above, but now we see that there is another eigenvalue at $\lambda = 3$. Let's compute the eigenvector associated to $\lambda = 3$. We know that $(A - 3I)\mathbf{v}_3 = 0$, so if $\mathbf{v}_3 = (v_1, v_2)^T$ we get

$$\begin{aligned}(A - 3I)\mathbf{v}_3 &= \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} -v_1 - 4v_2 \\ -v_1 - 4v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\end{aligned}$$

Both equations give the same information, so just looking at one of them we have $v_1 = -4v_2$. This means that any vector which looks like

$$\begin{bmatrix} -4v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix} v_2$$

will be an eigenvector for $\lambda = 3$. A simple choice might be to set $v_2 = 1$, so that $\mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. The keen student can check that

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$

Note that sometimes eigenvalues might not exist, for example, if we try to compute that eigenvalues of the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we get

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

which has no roots.

Proposition 3.21

A matrix A is invertible if and only if zero is not one of its eigenvalues.

Proof. If 0 is an eigenvalue, then $\det(A - \lambda I) = \det(A) = 0$, showing that A is not invertible. Similarly, if A is not invertible, $\det(A) = \det(A - 0 \cdot I) = 0$ showing that 0 is an eigenvalue. □

Example 3.22

Compute the eigenvalues and eigenvectors for the 3×3 matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Solution. We have

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{bmatrix}1-\lambda & -1 & 0\\ -1 & 2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{bmatrix}\\ &= (1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right]-(1-\lambda)\\ &= (1-\lambda)\left[(2-\lambda)(1-\lambda)-2\right]\\ &= (1-\lambda)\left[\lambda^2-3\lambda+2-2\right]\\ &= \lambda(1-\lambda)(\lambda-3).\end{aligned}$$

Hence our eigenvalues are $0,1,3$. When $\lambda=0$ we row reduce to find

$$\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} \xrightarrow{R_1+R_2\to R_2} \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & -1 & 1\end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}.$$

Hence if $\mathbf{v}=(v_1,v_2,v_3)$ then $v_1=v_2=v_3$. A nice choice is $(1,1,1)$. When $\lambda=1$ we have

$$\begin{bmatrix}0 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 0\end{bmatrix} \xrightarrow[R_1\leftrightarrow R_2]{-R_1+R_3\to R_3} \begin{bmatrix}-1 & 1 & -1\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Hence our eigenvector is $(1,0,-1)$. Finally, if $\lambda=3$ then

$$\begin{bmatrix}-2 & -1 & 0\\ -1 & -1 & -1\\ 0 & -1 & -2\end{bmatrix} \xrightarrow[R_2\leftrightarrow R_1]{-2R_2+R_1\to R_1} \begin{bmatrix}-1 & -1 & -1\\ 0 & 1 & 2\\ 0 & -1 & -2\end{bmatrix} \xrightarrow{R_2+R_2\to R_2} \begin{bmatrix}-1 & -1 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$$

yielding an eigenvector $(1,-2,1)$. ■

We could have chosen different eigenvectors by choosing different free parameters, but this just corresponds to a difference by a scalar multiple. We can verify this as follows:

Proposition 3.23

If $\mathbf{v}_\lambda$ is an eigenvalue for A with corresponding eigenvalue λ, then for any non-zero $c\in\mathbb{R}$, $c\mathbf{v}_\lambda$ is also an eigenvalue for A, with the same eigenvalue.

Proof. Let c be a non-zero real number. To check if $c\mathbf{v}_\lambda$ is an eigenvalue, we act on it by A to get

$$\begin{aligned}A(c\mathbf{v}_\lambda) &= cA\mathbf{v}_\lambda\\ &= c\lambda\mathbf{v}_\lambda && \text{since } A\mathbf{v}_\lambda=\lambda\mathbf{v}_\lambda\\ &= \lambda(c\mathbf{v}_\lambda)\end{aligned}$$

exactly as required. □

This is interesting, since it effectively means that all scalar multiples of an eigenvector are eigenvectors. This motivates us to define the following:

Definition 3.24

Let A be an $n \times n$ matrix with eigenvalue λ. We define the *eigenspace* of λ to be the set

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\}.$$

That is, the eigenspace of λ is the set of all eigenvalues of λ (plus the zero vector).

3.5 Diagonalization

Multiplying matrices is a lot of work, and occurs often in applications. For example, Markov processes used in machine learning and statistics requires one to compute $A^k\mathbf{v}_0$ for a matrix A and a vector $\mathbf{v}_0$, to determine how a system evolves statistically over time. In solving differential equations, one often has to take the matrix exponential e^A, which is defined as

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n + \cdots. \tag{3.2}$$

And countless more examples exist.

It is simple to exponentiate diagonal matrices, since

$$\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}^k = \begin{bmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{bmatrix}$$

so if we could somehow write our matrix A in terms of a diagonal, it would be easier to do these types of computations.

Definition 3.25

We say that an $n \times n$ matrix A is *diagonalizable* if there exists an $n \times n$ matrix P and a diagonal matrix D such that
$$A = PDP^{-1}.$$

If A is diagonalizable with $A = PDP^{-1}$ then

$$\begin{aligned} A^k = (PDP^{-1})^k &= \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{k\text{-times}} \\ &= PD\underbrace{(P^{-1}P)}_{I_n}D\underbrace{(P^{-1}P)}_{I_n}\cdots\underbrace{(P^{-1}P)}_{I_n}DP^{-1} \\ &= PD^kP^{-1}. \end{aligned}$$

How do we find D and P? The condition $A = PDP^{-1}$ is equivalent to $AP = PD$. Let $\mathbf{x}_1$ be the columns of P and $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ so that $AP = PD$ is equivalent to

$$A \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \end{bmatrix}.$$

That is, the columns of P are the eigenvectors of A, and the diagonal matrix consists of the eigenvalues of A. Using our previous examples, we showed that

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

had eigenvalues $\lambda = 3$ and $\lambda = -2$ with eigenvectors $\begin{bmatrix} -4 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ respectively. We set $D = \operatorname{diag}(3, -2)$ and

$$P = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{5} \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}.$$

We can check that this gives us the correct diagonalization by multiplying, to get

$$\begin{aligned} PDP^{-1} &= \frac{1}{5} \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 3 \\ -2 & -8 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 10 & -20 \\ -5 & -5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \\ &= A \end{aligned}$$

exactly as we suspected.

Theorem 3.26

A square $n \times n$-matrix A is diagonalizable if and only if it admits n-eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

is invertible.

Example 3.27

Diagonalize the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution. We first compute the eigenvalues:

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)(4-\lambda)(3-\lambda),$$

using the cofactor expansion on the first row. This determinant is zero precisely when $\lambda = 3$ (with multiplicity 2) and $\lambda = 4$. When $\lambda = 3$ our matrix becomes

$$\mathbf{0} = (A - 3I)\mathbf{v} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3v_1 + v_2 + 9v_3 \\ 0 \end{bmatrix}.$$

Let $v_2 = 3s$ and $v_3 = t$ so that $v_1 = s + 3t$ giving solution

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} t.$$

Our eigenvectors are thus $\mathbf{v}_1 = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}^T$. For $\lambda = 4$ we get

$$\mathbf{0} = (A - 4I)\mathbf{v} = \begin{bmatrix} -1 & 0 & 0 \\ -3 & 0 & 9 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -3v_1 + 9v_3 \\ -v_3 \end{bmatrix}$$

showing that $v_1 = v_3 = 0$ and allowing v_3 to be free. Hence $\mathbf{v}_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. Our matrix P is

$$P = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \text{whose inverse is} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix},$$

so

$$\begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix}.$$

You can double check the result by multiplying these matrices. ■

Note that diagonalizability and invertibility are not related. For example, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is certainly invertible – it has non-zero determinant – but I claim it is not diagonalizable. To see that this is the case, the lone eigenvalue of A is 1, but solving the system $A\mathbf{v} = \mathbf{v}$ yields

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_2 \end{bmatrix}.$$

Hence $v_2 = 0$ and v_1 is free, so the solutions to this are of the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} s, \qquad \text{for } s \in \mathbb{R}.$$

Since this matrix has only a single eigenvalue, we cannot form the P matrix, and so conclude that the matrix is not diagonalizable. By the same token, the matrix

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has no (real) eigenvalues, and hence cannot be diagonalized.

Example 3.28

If A is the matrix from Example 3.27, compute A^k for any $k \in \mathbb{N}$, and e^A where the exponential is as defined in (3.2).

Solution. We know the diagonalization of A from Example 3.27 is

$$\begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix}.$$

If $k \in \mathbb{N}$ then

$$\begin{aligned} \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}^k &= \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 3^k & 3^{k+1} & 0 \\ 3^{k+1} & 0 & 4^k \\ 0 & 3^k & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 3^k & 0 & 0 \\ 3^{k+1} - 3(4^k) & 4^k & -3^{k+2} + 9(4^k) \\ 0 & 0 & 3^k \end{bmatrix}. \end{aligned}$$

If $D = \operatorname{diag}(3,3,4)$ then it's easy to check that $e^D = \operatorname{diag}(e^3, e^3, e^4)$, and

$$\begin{aligned} e^A = e^{PDP^{-1}} &= I + PDP^{-1} + \frac{(PDP^{-1})^2}{2!} + \frac{(PDP^{-1})^3}{3!} + \cdots + \frac{(PDP^{-1})^n}{n!} + \cdots + \\ &= I + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots + \frac{PD^nP^{-1}}{n!} + \cdots + \\ &= P\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots + \frac{D^n}{n!} + \cdots\right)P^{-1} \\ &= Pe^DP^{-1}. \end{aligned}$$

Hence

$$\begin{aligned}
e^A &= \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^3 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix} \\
&= \begin{bmatrix} e^3 & e^3 & 0 \\ 3e^3 & 0 & e^4 \\ 0 & e^3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ -3 & 1 & 9 \end{bmatrix} \\
&= \begin{bmatrix} e^3 & 0 & -2e^3 \\ 3(e^3 - e^4) & e^4 & 9(e^4 - e^3) \\ 0 & 0 & e^3 \end{bmatrix}.
\end{aligned}$$

This computation would have been near impossible without diagonalization. ■

3.6 Applications

Eigenvalues and eigenvectors are inherent to a matrix and therefore appear often whenever a matrix is used in applications. The example we introduce below is that of linear dynamical systems. These are worth studying on their own as one often studies *non-linear* dynamical systems by looking at their linear approximations.

Consider the following scenario. You have a population of rabbits and foxes, where the rabbits have unlimited resources to grow, and are predated upon only by foxes. Similarly, foxes feed only upon rabbits, and die of natural causes. The populations are dependent upon one another in the following way:

- There are initially 1000 rabbits and 50 foxes,
- The number of rabbits will double each year,
- The foxes eat on average 12 rabbits per year,
- The fox population increases with more rabbits, proportional to 10% the number of rabbits,
- 20% of the foxes die each year.

If r_k and f_k are the number of rabbits and foxes at the end of year k respectively, then

$$\begin{aligned}
r_{k+1} &= 2r_k - 12f_k \\
f_{k+1} &= 0.1r_k - 0.2f_k.
\end{aligned}$$

Let $\mathbf{p}_k = \begin{bmatrix} r_k & f_k \end{bmatrix}^T$ and set

$$A = \begin{bmatrix} 2 & -12 \\ 0.1 & -0.2 \end{bmatrix}$$

so that $\mathbf{p}_{k+1} = A\mathbf{p}_k$, with $\mathbf{p}_0 = \begin{bmatrix} 1000 & 50 \end{bmatrix}^T$. Equivalently, $\mathbf{p}_{k+1} = A^{k+1}\mathbf{p}_0$. This system is plotted in Figure 8. Notice that both populations initially increase before finding an equilibrium.

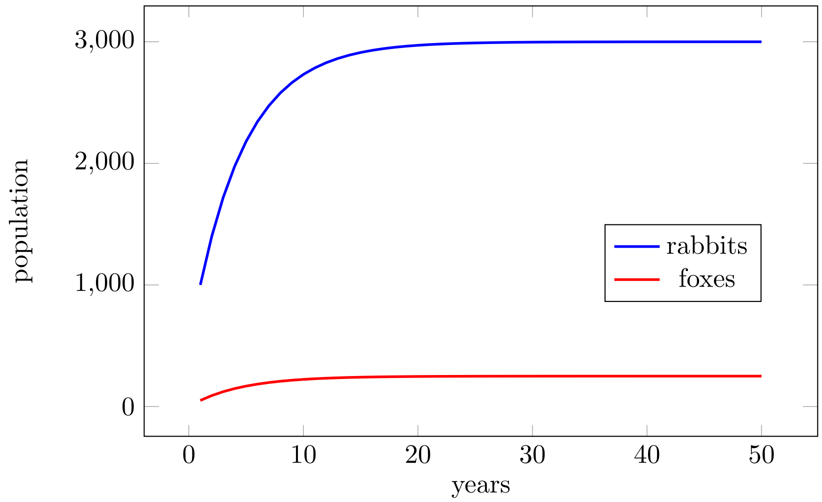

Figure 8: The population of foxes and rabbits according to the dynamics above.

How can we analyze this system absent a computer? Well, if we diagonalize A as $A = PDP^{-1}$, with $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ the eigenvalues of A, then

$$\mathbf{p}_k = A^k \mathbf{p}_0 = PD^k P^{-1} \mathbf{p}_0 = PD^k \mathbf{b}$$

where $\mathbf{b} = P^{-1}\mathbf{p}_0$. Let $\mathbf{x}_k$ be the columns of P. Expanding out the right hand side, using the fact that D^k is still diagonal, we get

$$\mathbf{p}_k = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b_1 \lambda_1^k \mathbf{x}_1 + b_2 \lambda_2^k \mathbf{x}_2 + \cdots + b_n \lambda_n^k \mathbf{x}_n. \tag{3.3}$$

Definition 3.29

Let A be an $n \times n$ square matrix with eigenvalues $\lambda_1, \ldots, \lambda_k$. The *dominant eigenvalue* λ_{dom} is the eigenvalue of greatest magnitude; that is, $|\lambda_{\text{dom}}| \geq |\lambda_j|$ for all $j = 1, \ldots, k$.

Suppose that λ_1 is the dominant eigenvalue of A. By factoring the right hand side of (3.3) by λ_1^k, we get

$$\mathbf{p}_k = \lambda_1^k \left(b_1 \mathbf{x}_1 + b_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k \mathbf{x}_2 + \cdots + b_n \left(\frac{\lambda_n}{\lambda_1} \right)^k \mathbf{x}_n \right).$$

Since λ_1 is dominant, we know $|\lambda_j / \lambda_k| < 1$, which tend to zero as $k \to \infty$, hence as k becomes large, we get

$$\mathbf{p}_k \approx b_1 \lambda_1^k \mathbf{x}_1.$$

This is indeed the case for our example above. It is a bit of work, but one can show that the diagonalization of A is (with rounding)

$$\begin{bmatrix} 2 & -12 \\ 0.1 & -0.2 \end{bmatrix} = \begin{bmatrix} 0.99654 & 0.995037 \\ 0.083045 & 0.099504 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} 6.0208 & -60.2080 \\ -5.0249 & 60.2993 \end{bmatrix}$$

The dominant eigenvalue is $\lambda_{\text{dom}} = 1$, and

$$\mathbf{x}_1 = \begin{bmatrix} 0.99654 \\ 0.083045 \end{bmatrix}, \qquad b_1 = 3010.4,$$

so that

$$\mathbf{p}_k \approx 3010.4 \begin{bmatrix} 0.99654 \\ 0.083045 \end{bmatrix} = \begin{bmatrix} 3000 \\ 250 \end{bmatrix}$$

which is what we see in Figure 8. In my mind, what is more interesting is that we can use this to predict how the initial population rations will affect long term evolution. Note that $b_1 = 6.0208r_0 - 60.2080f_0$, so if $r_0/f_0 = 10$ then $b_1 = 0$ and both populations will go extinct.

4 Vector Spaces

We now being a somewhat more abstract study of linear algebra. The general idea is that it's possible to describe matrices and linear transformations in more than just the standard Cartesian coordinates.

4.1 Vectors

The physicists define a vector as something with direction and magnitude. Strictly speaking, this is wrong. I am therefore torn as to how to define a vector for you, since I do not want to give you the wrong definition, yet the curriculum does not call for me to give you the correct definition.

For now, let's define define three dimensional real vectors. Any element $\mathbf{v} \in \mathbb{R}^3$ is a three dimensional real vector, say $\mathbf{v} = (x_0, y_0, z_0)^T$. As I mentioned in Section 1.4.2, you can think of such a point as an arrow in three dimensional space, spanning from the origin $O = (0, 0, 0)$ to $\mathbf{v}$.

One of the most important ideas in linear algebra is that the same vector/matrix can be written in many different ways, depending on what coordinate system you use. For this reason, there should be a coordinate independent way of thinking about vectors, vector addition, and scalar multiplication. We saw this already in Figure 2, but the idea is that to add two vector, we add them "tip-to-tail," while scalar multiplication is done by scaling the vector.

Definition 4.1

If $\mathbf{v} = (x_0, y_0, z_0) \in \mathbb{R}^3$, we define the *norm* of $\mathbf{v}$ as

$$\|\mathbf{v}\| = \sqrt{x_0^2 + y_0^2 + z_0^2}.$$

The norm of $\mathbf{v}$ is precisely the length of the arrow from O to $\mathbf{v}$. This can be seen in Figure 10. Additionally, recall that

$$\mathbf{v} \cdot \mathbf{v} = x_0^2 + y_0^2 + z_0^2$$

so that $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. This will be discussed in greater detail when we reexamine the dot product.

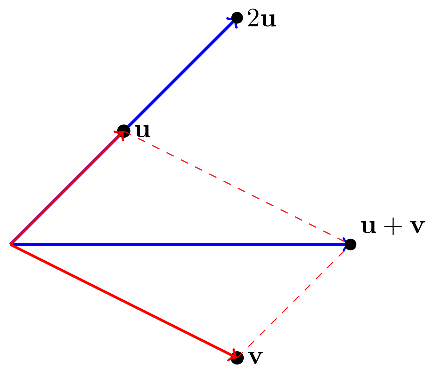

Figure 9: We add two vectors by adding them "tip-to-tail," and scalar multiplication is done by scaling.

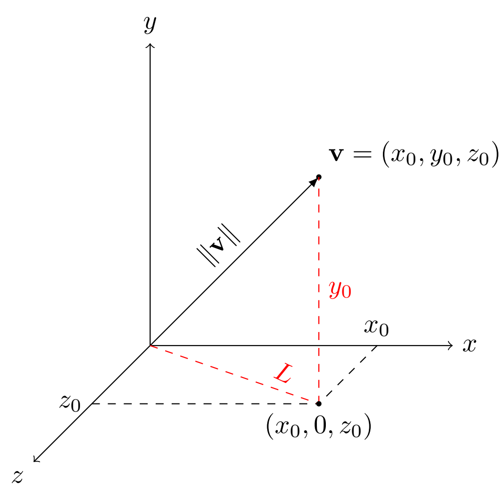

Figure 10: The length of the line L is $\sqrt{x_0^2 + z_0^2}$, so that the triangle formed by the origin, $(x_0, 0, z_0)$ and $\mathbf{v}$ has length $\|\mathbf{v}\| = \sqrt{L^2 + y_0^2} = \sqrt{x_0^2 + y_0^2 + z_0^2}$.

Example 4.2

A point on the sphere of radius 1 is given by

$$\mathbf{v} = (\cos(\theta)\sin(\phi), \cos(\theta)\cos(\phi), \sin(\theta))$$

for $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. What should $\|\mathbf{v}\|$ be? Show that this is correct.

Solution. Since $\mathbf{v}$ lives on the unit sphere centred at the origin, it should have length 1. We compute the norm to check this. Note that carrying around the square root sign is messy, so we'll show that $\|\mathbf{v}\|^2 = 1$ which is equivalent to $\|\mathbf{v}\| = 1$.

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \cos^2(\theta)\sin^2(\phi) + \cos^2(\theta)\cos^2(\phi) + \sin^2(\theta) \\
&= \cos^2(\theta)\left[\sin^2(\phi) + \cos^2(\phi)\right] + \sin^2(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta) && \text{by Pythagoras} \\
&= 1 && \text{also by Pythagoras,}
\end{aligned}$$

which is what we wanted to show. ■

Proposition 4.3

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

1. $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = 0$ (Non-degeneracy),
2. $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$ (Homogeneity),
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (Triangle Inequality).

We say that $\mathbf{u} \in \mathbb{R}^3$ is a *unit vector* if $\|\mathbf{u}\| = 1$. For example, the standard basis vectors

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)$$

are all unit vectors. A non-trivial example is $\mathbf{u} = (2/3, 1/3, 2/3)$, which is also a unit vector as

$$\|\mathbf{u}\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{1} = 1.$$

We often use unit vector to describe the direction of a vector. If $\mathbf{v} \in \mathbb{R}^3$, let $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$. This is a unit vector, since

$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1,$$

but since it is just a scalar multiple of $\mathbf{v}$, it points in the same direction.

Our text introduces something called *geometric vectors*. These are vectors whose tails need not start at the origin. This is a non-standard name, and one often does not make the distinction between vectors and geometric vectors. However, if $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ we can define the vector $\overrightarrow{PQ}$ as the vector whose tail is P and whose head is Q. We can translate the base of this vector to the origin by recognizing that $\overrightarrow{PQ} = \mathbf{Q} - \mathbf{P}$, so we associate $\overrightarrow{PQ}$ with $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$, from which it's easy to see that

$$\left\|\overrightarrow{PQ}\right\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}.$$

This tells us that given two vectors $\mathbf{u}$ and $\mathbf{v}$, the distance between $\mathbf{u}$ and $\mathbf{v}$ is precisely $\|\mathbf{u} - \mathbf{v}\|$. In general, we will identify (think of as being equal) any two vectors with the same length length and direction. In this sense, $\overrightarrow{PQ}$ is the same as $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$.

Definition 4.4

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ are said to be *parallel* if they point in the same direction.

We cannot yet show it, but two vectors are parallel if and only if one is a scalar multiple of the other; that is, there exists a non-zero $\alpha \in \mathbb{R}$ such that $\mathbf{u} = \alpha\mathbf{v}$. We can use this idea to construct lines. For example, given a direction $\mathbf{d}$, the line through the origin in the direction of $\mathbf{d}$ is the collection of all vectors parallel to $\mathbf{d}$. Hence we can parameterize that line as $L(t) = t\mathbf{d}$ for $t \in \mathbb{R}$. On the other hand, to get a line passing through the point $\mathbf{p}_0$, we simply translate everything by adding $\mathbf{p}_0$, giving

$$L(t) = \mathbf{p}_0 + t\mathbf{d}.$$

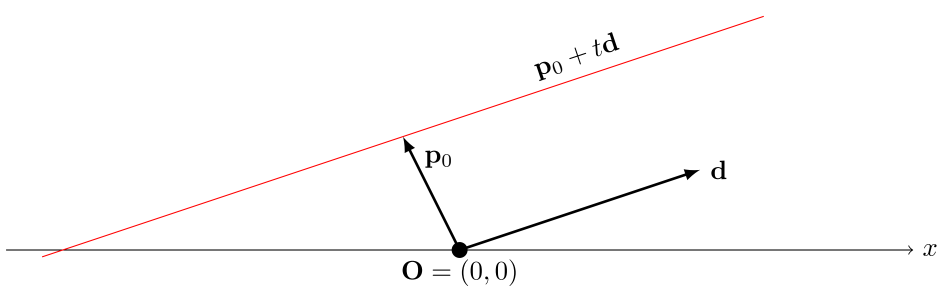

Figure 11: The line parameterized by $\mathbf{p}_0 + t\mathbf{d}$ can be seen as all scalar multiples of $\mathbf{d}$ translated by $\mathbf{p}_0$.

Example 4.5

Parameterize the line passing through the points $P = (1, 0, 1)$ and $Q = (-2, 1, 0)$.

Solution. We know that our line passes through either of the points P or Q, so we can take either of these for $\mathbf{p}_0$. Let's choose $\mathbf{p}_0 = P$. Now we need direction vector of this line, which is given by $\overrightarrow{PQ}$, with coordinates

$$\mathbf{d} = \overrightarrow{PQ} = (-3, 1, -1).$$

Hence the line is parameterized by

$$\mathbf{p}_0 + t\mathbf{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-3t \\ t \\ 1-t \end{bmatrix}. \qquad \blacksquare$$

Example 4.6

Let L_1 be the line we found in Example 4.5, and L_2 be the line parameterized by $(1+6s, 1-4s, 1+2s)$. Determine where, if ever, L_1 intersects L_2.

Solution. We want to find t and s such that

$$\begin{bmatrix} 1-3t \\ t \\ 1-t \end{bmatrix} = \begin{bmatrix} 1+6s \\ 1-4s \\ 1+2s \end{bmatrix} \qquad \text{or equivalently} \qquad \begin{bmatrix} 6s+3t \\ -4s-t \\ 2s+t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We can write this is the linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 6 & 3 \\ -4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Row reducing the augmented A gives

$$\left[\begin{array}{cc|c} 6 & 3 & 0 \\ -4 & -1 & -1 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{(1/3)\mathrm{R}_1 \to \mathrm{R}_1} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -1 & -1 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow[(-1)\mathrm{R}_1 + \mathrm{R}_3 \to \mathrm{R}_3]{2\mathrm{R}_1 + \mathrm{R}_2 \to \mathrm{R}_2} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

So that $t = -1$ and $2s + t = 0$, which can be solve to yield $(s,t) = (1/2, -1)$. Indeed, if we plug this into our lines, we get

$$L_1 : \begin{bmatrix} 1-3t \\ t \\ 1-t \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \qquad L_2 : \begin{bmatrix} 1+6s \\ 1-4s \\ 1+2s \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix},$$

so the lines intersect at $(4, -1, 2)$. $\blacksquare$

4.2 Dot Product and Projections

4.2.1 Dot Product Revisited

Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3)$ be vector in $\mathbb{R}^3$. We've seen that their dot product can be computed as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

It's important to keep in mind that this 'product' between two vectors returns a real number, *not* another vector. The dot product satisfies the following properties:

Proposition 4.7

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
2. $\mathbf{u} \cdot \mathbf{0} = \mathbf{0}$,
3. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$,
4. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
5. $(\alpha \mathbf{u}) \cdot \mathbf{v} = \alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha \mathbf{v})$.

The geometric insight behind the dot product can be gleaned with the following:

Proposition 4.8

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$, where θ is the angle subtended by $\mathbf{u}$ and $\mathbf{v}$ in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

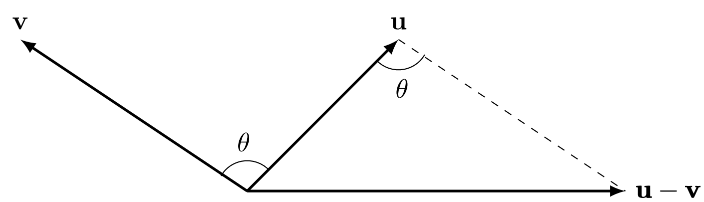

Figure 12: The triangle formed by the origin, $\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$. We can use the law of cosines to determine the length of $\mathbf{u} - \mathbf{v}$.

Proof. Consider the triangle in Figure 12, so that by the cosine law we have

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta).$$

On the other hand, we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

Equating both equations gives $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$ as required. □

Proposition 4.8 tells us many things. The first is that $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* – that is, they form an angle of $\theta = \pi/2$ – precisely when $\mathbf{u} \cdot \mathbf{v} = 0$. For example, the vectors $\mathbf{u} = (1, 0, 2)$ and $\mathbf{v} = (-2, 1, 1)$ are orthogonal, since

$$\mathbf{u} \cdot \mathbf{v} = (1 \times -2) + (0 \times 1) + (2 \times 1) = -2 + 2 = 0.$$

It also tells us that the angle between two vectors can be determined as

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right). \tag{4.1}$$

For example, the angle between $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (1, 1, 0)$ is

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

Since (4.1) holds for all $\mathbf{u}, \mathbf{v}$, and the domain of arccosine is $[-1, 1]$, this suggests that

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|,$$

which is called the *Cauchy-Schwarz inequality.*

Another application is the geometric interpretation of the dot product. Suppose that $\mathbf{u}$ is a unit vector, so $\|\mathbf{u}\| = 1$. If $\mathbf{v}$ is any other vector, then

$$\mathbf{v} \cdot \mathbf{u} = \|\mathbf{v}\|\|\mathbf{u}\| \cos(\theta) = \|\mathbf{v}\| \cos(\theta).$$

Look at Figure 13. The value $\|\mathbf{v}\| \cos(\theta)$ is precisely the length of the *projection* of $\mathbf{v}$ onto $\mathbf{u}$.

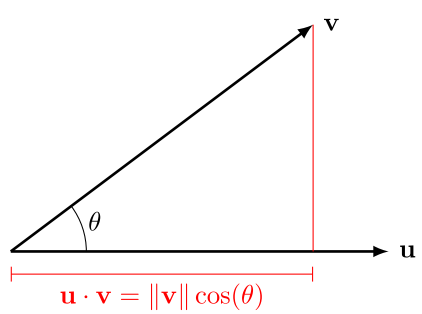

Figure 13: If $\mathbf{u}$ is a unit vector ($\|\mathbf{u}\| = 1$) then $\mathbf{u} \cdot \mathbf{v}$ is the length of the $\mathbf{v}$ when projected onto $\mathbf{u}$.

4.2.2 Projections onto Lines

The idea of a projection is to extend the idea shown in Figure 13. Namely, suppose that you are given two vectors, $\mathbf{u}$ and $\mathbf{v}$. We should be able to write $\mathbf{v}$ as the sum $\mathbf{v} = \mathbf{u}_p + \mathbf{u}_o$, where $\mathbf{u}_p$ is parallel to $\mathbf{u}$, and $\mathbf{u}_o$ is orthogonal to $\mathbf{u}$ – see Figure 14. We define the *projection* of $\mathbf{v}$ onto $\mathbf{u}$ as $\mathbf{u}_p$; that is

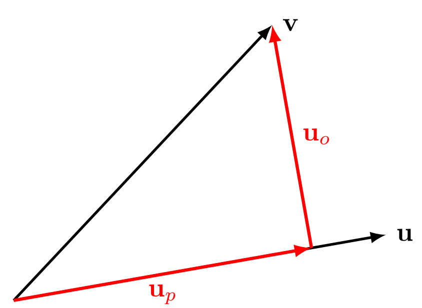

Figure 14: The projection of $\mathbf{v}$ onto $\mathbf{u}$.

Definition 4.9

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we define the projection of $\mathbf{v}$ onto $\mathbf{u}$, denoted $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$, as

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}. \tag{4.2}$$

In this case, the orthogonal component $\mathbf{u}_o$ is $\mathbf{u}_o = \mathbf{v} - \operatorname{proj}_{\mathbf{u}}(\mathbf{v})$.

Note that when $\mathbf{u}$ is a unit vector, the projection is $\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = (\mathbf{u} \cdot \mathbf{v})\mathbf{u}$, which is precisely what we found above. To see why the extra $\|\mathbf{u}\|^2$ term comes to play, rewrite the projection as

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \mathbf{v} \right) \frac{\mathbf{u}}{\|\mathbf{u}\|} = (\hat{\mathbf{u}} \cdot \mathbf{v})\, \hat{\mathbf{u}}, \qquad \text{where } \hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Thus when $\mathbf{u}$ is not a unit vector, the $\|\mathbf{u}\|^2$ term can be redistributed to give $\hat{\mathbf{u}} = \mathbf{u}/\|\mathbf{u}\|$ which is a unit vector.

Example 4.10

Let $\mathbf{u} = (2, 1, -4)$ and $\mathbf{v} = (1, -1, 1)$. Determine $\operatorname{proj}_{\mathbf{u}}(\mathbf{v})$.

Solution. Let's compute the dot products and norms separately:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (2, 1, 4) \cdot (1, -1, 1) = 2 - 1 + 4 = 5, \\
\|\mathbf{u}\|^2 &= 4 + 1 + 16 = 21,
\end{aligned}$$

Putting this all together and substituting into (4.2) yields

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{5}{21}\mathbf{u} = \left(\frac{10}{21}, \frac{5}{21}, -\frac{20}{21} \right).$$

■

Example 4.11

Consider the line between the points $P = (1, 2, -1)$ and $Q = (2, 0, 3)$. Find the shortest distance from this line to the point $R = (1, 1, 1)$.

Solution. We begin by finding the parametric equation for the line through P and Q, which is given by

$$L(t) = P + (Q - P)t = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} t = \begin{bmatrix} 1 + 2t \\ 2 - 2t \\ -1 + 4t \end{bmatrix}.$$

Now the shortest distance from this line to $R = (1, 1, 1)$ will be the orthogonal component of the projection of $\overrightarrow{PR}$ onto the slope of the line. From this we get

$$\overrightarrow{PR} = (0, -1, 2),$$

and the projection of this onto the slope $\mathbf{m} = (2, -2, 4)$ is

$$\operatorname{proj}_{\mathbf{m}} \overrightarrow{PR} = \frac{\mathbf{m} \cdot \overrightarrow{PR}}{\|\mathbf{m}\|^2}\mathbf{m} = \frac{6}{24}\mathbf{m} = \frac{1}{4}(2, -2, 4).$$

The orthogonal component is then

$$\mathbf{m}_o = \overrightarrow{PR} - \operatorname{proj}_{\mathbf{m}} \overrightarrow{PR} = (0, -1, 2) - \frac{1}{4}(2, -2, 4) = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right),$$

which has length

$$\|\mathbf{m}_o\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}. \qquad \blacksquare$$

4.3 Cross Product and Planes

A line in $\mathbb{R}^3$ is specified by two points. A plane in $\mathbb{R}^3$ on the other hand can be specified by three points, though this is not practical for writing down the plane. On the other hand, we can also use a point and a line: the line is orthogonal to the plane, and the point describes where the plane passes through that line. Hence given a vector $\mathbf{n}$, thought of as the line, and a point $\mathbf{x}_0$, the equation of a plane in $\mathbb{R}^3$ is given by

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0.$$

We call $\mathbf{n}$ the *normal vector* to the plane. If $\mathbf{n} = (a, b, c)$, $\mathbf{x} = (x, y, z)$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ then this becomes

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d, \qquad \text{where} \qquad d = ax_0 + by_0 + cz_0.$$

Example 4.12

Find the equation of the plane through $P_0 = (1, 1, 1)$ with normal vector $\mathbf{n} = (-1, 0, 1)$.

Solution. Using our equation above, the equation of the plane is $\mathbf{n}(\mathbf{x} - P_0) = 0$, which gives

$$-1(x - 1) + 0(y - 1) + 1(z - 1) = 0 \quad \text{or} \quad -x + z = 0. \qquad \blacksquare$$

Two planes in $\mathbb{R}^3$ can either not intersect at all, intersect at a single point, or intersect in a plane. We in fact already know how to find the intersection of two planes.

Example 4.13

Consider two planes, the first with normal vector $\hat{\mathbf{n}}_1 = (1, 3, 5)$ passing through $P_1 = (8, 0, -2)$, and and the second with normal vector $\hat{\mathbf{n}}_2 = (2, 5, 9)$ passing through the point $P_2 = (6, -3, 0)$. Find the intersection of these two planes.

Solution. The planes themselves can be written as

$$\begin{aligned}(1,3,5)\cdot(x-8,y,z+2)=0 \quad&\Rightarrow\quad x+3y+5z=-2\\(2,5,9)\cdot(x-6,y+3,z)=0 \quad&\Rightarrow\quad 2x+5y+9z=-3.\end{aligned}$$

The intersection of these two planes is the collection of (x,y,z) which lie in both planes simultaneously; namely, is a solution to the linear system given by both plane equations. By now we're experts at solving this sort of problem, giving

$$\left[\begin{array}{ccc|c}1&3&5&-2\\-2&-5&-9&3\end{array}\right]\xrightarrow{\text{RREF}}\left[\begin{array}{ccc|c}1&0&2&1\\0&1&1&-1\end{array}\right].$$

This gives the solution

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}+\begin{bmatrix}-2\\-1\\1\end{bmatrix}s,$$

which we recognize as the equation of a line in $\mathbb{R}^3$. ■

Example 4.14

Find the shortest distance from the point $P=(2,3,0)$ and the plane $5x+y+z=1$, and the point on the plane where this distance is realized.

Solution. The plane has a normal vector of $\mathbf{n}=(5,1,1)$, and the point $P_0=(0,0,1)$ is certainly in the plane, as it satisfies the equation. The vector from P_0 to P is $\mathbf{u}=\overrightarrow{P_0P}=(2,3,-1)$, and the shortest distance from P to the plane will be the length of the projection of $\mathbf{u}$ onto $\mathbf{n}$, which gives

$$\operatorname{proj}_{\mathbf{n}}(\mathbf{u})=\frac{\mathbf{u}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}=\frac{10+3-1}{25+1+1}\mathbf{n}=\frac{12}{27}\mathbf{n}.$$

The length of this is

$$\left\|\frac{12}{27}\mathbf{n}\right\|=\frac{12}{27}\sqrt{27}=\frac{12}{\sqrt{27}}.$$

The point on the plane which realizes this distance will be

$$P-(12/27)\mathbf{n}=\begin{bmatrix}2\\3\\0\end{bmatrix}-\frac{12}{27}\begin{bmatrix}5\\1\\1\end{bmatrix}=\frac{1}{9}\begin{bmatrix}-2\\23\\-4\end{bmatrix}$$

which you can check does indeed satisfy $5x+y+z=1$. ■

Given that you are forced to use three points to determine the plane, one strategy is to find a vector which is normal to two of those vectors, and reduce it to the equation of a plane above. This is done using cross-products.

Definition 4.15

If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ then their *cross-product* is

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, x_2 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

A useful technique for remembering the cross-product is to abuse the determinant. Suppose that $\hat{\imath} = \mathbf{e}_1, \hat{\jmath} = \mathbf{e}_2, \hat{k} = \mathbf{e}_3$, then

$$\mathbf{x} \times \mathbf{y} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Example 4.16

Determine $\mathbf{e}_1 \times \mathbf{e}_2$.

Solution. Intuitively, we know that any scalar multiple of $\mathbf{e}_3$ is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$, so we expect something of this form out. Computing the cross product gives

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (0, 0, 1)$$

which is what we expected. ■

Proposition 4.17

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$.

1. If $\mathbf{u}, \mathbf{v} \neq 0$, then $\mathbf{v} \times \mathbf{w} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel; that is, there exists some $\lambda \neq 0$ such that $\mathbf{u} = \lambda \mathbf{v}$.
2. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Example 4.18

Find the equation of the plane through the points $P = (4, 0, 5), Q = (2, 0, 1)$ and $R = (1, -1, 2)$.

Solution. The vectors $\mathbf{u} = \overrightarrow{QP} = (2, 0, 4)$ and $\mathbf{v} = \overrightarrow{QR} = (-1, -1, 1)$ both lie in the plane, so we can compute the normal to these vectors as

$$\mathbf{n} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 4 \\ -1 & -1 & 1 \end{bmatrix} = (4, -6, -2).$$

We can use any of P, Q, R for the point through which the plane must pass, say Q, so the equation of the plane is

$$\mathbf{n} \cdot (\mathbf{x} - Q) = 0 \quad \Rightarrow \quad 4(x-2) - 6y - 2(z-1) = 0$$

or equivalently $4x - 6y - 2z = 6$. A quick check shows that P, Q, R all satisfy this equation. ■

Theorem 4.19

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, then

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$,
2. $\mathbf{u} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{u}$,
3. $(\alpha \mathbf{u}) \times \mathbf{v} = \alpha(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (\alpha \mathbf{v})$,
4. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$,
5. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det[\mathbf{uvw}]$.

Example 4.20

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$. Show that $\mathbf{v} - \mathbf{w}$ is orthogonal to

$$\mathbf{z} = (\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \times \mathbf{w}) + (\mathbf{w} \times \mathbf{u}).$$

Solution. We need to show that $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{z} = 0$. Note that

$$\begin{aligned}
(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{u} \times \mathbf{v}) &= \det[(\mathbf{v} - \mathbf{w})\mathbf{uv}] = \det[\mathbf{vuv}] - \det[\mathbf{wuv}] = -\det[\mathbf{wuv}] \\
(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w}) &= \det[(\mathbf{v} - \mathbf{w})\mathbf{vw}] = \det[\mathbf{vvw}] - \det[\mathbf{wvw}] = 0 \\
(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{w} \times \mathbf{u}) &= \det[(\mathbf{v} - \mathbf{w})\mathbf{wu}] = \det[\mathbf{vwu}] - \det[\mathbf{wwu}] = \det[\mathbf{vwu}].
\end{aligned}$$

Moreover, $\det[\mathbf{vwu}] = -\det[\mathbf{wvu}] = \det[\mathbf{wuv}]$, so adding all three terms gives zero, as required. ■

Theorem 4.21: Lagrange's Identity

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ then

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

Using the fact that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$ this means that

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2. \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2(\theta) \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \left(1 - \cos^2(\theta)\right) \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2(\theta)
\end{aligned}$$

so that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)$. This is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Example 4.22

Find the area of the triangle whose vertices lie at $P = (5, 2, 2), Q = (1, 0, -1)$, and $R = (-3, 1, 2)$.

Solution. Note that a triangle is half a parallelogram, so by finding the area of the corresponding parallelogram we will be done. Choose one of the points to serve as the origin, say Q, in which case

$$\mathbf{u} = \overrightarrow{QP} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v} = \overrightarrow{QR} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}.$$

The parallelogram spanned by these has area $\|\mathbf{u} \times \mathbf{v}\|$, which we can compute as

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 2 & 3 \\ -4 & 1 & 3 \end{bmatrix} = (3, -24, 12),$$

so $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{729} = 27$, and the area of the triangle is $27/2$. ■

So the cross product gives the area of a parallelogram. The scalar triple product gives the area of the parallelepiped, as follows:

Theorem 4.23

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ then

$$|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| = |\det[\mathbf{uvw}]|$$

and is the area of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Example 4.24

Find the area of the parallelepiped spanned by the vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Solution. We first compute $\mathbf{v} \times \mathbf{w}$, which yields

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = (1, -4, 2).$$

Dotting against $\mathbf{u}$ gives

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 1 - 4 + 2 = -1,$$

so the area of the parallelepiped is 1. ■

4.4 Subspaces

When you think of a line $\mathbf{p} + t\mathbf{d}$ in $\mathbb{R}^3$, there is a sense in which this is just a copy of $\mathbb{R}$ sitting inside $\mathbb{R}^3$. Similarly, a plane $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ is a copy of $\mathbb{R}^2$ sitting inside $\mathbb{R}^3$. However, if we want these copies of $\mathbb{R}$ and $\mathbb{R}^2$ to behave well with vector addition and scalar multiplication, we must impose additional restrictions.

For example, consider the plane $P = \left\{(x,y,z) \in \mathbb{R}^3 : z = 1\right\}$ sitting in $\mathbb{R}^3$. The vectors $\mathbf{u} = (0,0,1)$ and $\mathbf{v} = (1,1,1)$ are both in this plane, but $\mathbf{u}+\mathbf{v} = (1,1,2)$ is not in P, nor is $\lambda u = (0,0,\lambda)$ for any $\lambda \neq 1$. This leads us the definition of a subspace.

Definition 4.25

A set $S \subseteq \mathbb{R}^3$ is a *subspace* of $\mathbb{R}^3$ if

1. $\mathbf{0} \in S$ (zero vector),
2. For every $\mathbf{u}, \mathbf{v} \in S$ we have $\mathbf{u} + \mathbf{v} \in S$ (closed under addition),
3. For every $\mathbf{u} \in S$ and $\lambda \in \mathbb{R}$, $\lambda \mathbf{u} \in S$ (closed under scalar multiplication).

The sets $\{0\}$ and $\mathbb{R}^3$ are both subspaces of $\mathbb{R}^3$. Any subspace which is not one of these is called a *proper subspace*.

Example 4.26

Consider the set
$$S = \left\{(x,y,z) \in \mathbb{R}^3 : x = y\right\}.$$
Determine whether S is a subspace of $\mathbb{R}^3$.

Solution. The zero vector $\mathbf{0} = (0,0,0)$ satisfies $x = y$, so $\mathbf{0} \in S$. Now suppose $\mathbf{u}, \mathbf{v} \in S$, say with
$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_1 \\ u_2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_1 \\ v_2 \end{bmatrix},$$
and $\lambda \in \mathbb{R}$. The sum $\mathbf{u} + \mathbf{v}$ and λu are given by
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}, \qquad \lambda \mathbf{u} = \begin{bmatrix} \lambda u_1 \\ \lambda u_1 \\ \lambda u_2 \end{bmatrix}.$$
In each case the x- and y- coordinates are identical, so both are in S. Hence S is a subspace of $\mathbb{R}^3$. ■

Example 4.27

Let A be a 3×3 matrix with non-zero eigenvalue λ. In Definition 3.24 we defined the eigenspace
$$E_\lambda = \left\{\mathbf{v} \in \mathbb{R}^3 : A\mathbf{v} = \lambda \mathbf{v}\right\}.$$
Show that the eigenspace is a subspace of $\mathbb{R}^3$.

Solution. We start with the $\mathbf{0}$-vector. Since $A\mathbf{0} = \lambda\mathbf{0} = \mathbf{0}$, we know $\mathbf{0} \in E_\lambda$. Suppose then that $\mathbf{u}, \mathbf{v} \in E_\lambda$, so that

$$A\mathbf{v} = \lambda\mathbf{v}, \qquad A\mathbf{u} = \lambda\mathbf{u}.$$

Note that

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + \mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u} + \mathbf{v})$$

so that $\mathbf{u} + \mathbf{v}$ is also in E_λ. Finally, let $\alpha \in \mathbb{R}$ be any real number. We have

$$A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha\lambda\mathbf{u} = \lambda(\alpha\mathbf{u}),$$

so $\alpha\mathbf{u} \in E_\lambda$, showing that E_λ is closed under scalar multiplication. With all three conditions satisfied, we conclude E_λ is a subspace. ■

Example 4.28

Determine whether the set $S = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 1\right\}$ is a subspace of $\mathbb{R}^3$.

Solution. This set satisfies none of the three properties, but perhaps the simplest to see is that it does not contain the $\mathbf{0}$-vector. Indeed, $\mathbf{0} = (0, 0, 0)$ so here $x^2 + y^2 = 0 \neq 1$. ■

Definition 4.29

Let A be an $m \times n$ matrix. The *null space* is the set

$$\operatorname{null}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

while the *image space* is

$$\operatorname{image}(A) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

Remark 4.30 I disagree with these naming conventions. The null space is sometimes called the *kernel*, and the image space is sometimes called the *column space*. Good practice is to either call these the null space and column space, or the kernel and image, but not to mix them. The different names comes from whether you are thinking of the matrix A by itself (null space/column space), or the corresponding linear transformation $T_A(\mathbf{x}) = A\mathbf{x}$ (kernel/image).

Proposition 4.31

The null space and image space of a matrix are both subspaces.

Proof. Let A be an $m \times n$ matrix, and let's start with the null space.

Certainly $A\mathbf{0} = \mathbf{0}$, so that $\mathbf{0} \in \operatorname{null}(A)$. Suppose that $\mathbf{x}, \mathbf{y} \in \operatorname{null}(A)$, so that $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. But then

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $\mathbf{0} \in \operatorname{null}(A)$. Similarly,

$$A(\alpha \mathbf{x}) = \alpha A\mathbf{x} = \alpha \mathbf{0} = \mathbf{0}$$

so $\alpha \mathbf{x} \in \operatorname{null}(A)$. We conclude $\operatorname{null}(A)$ is a subspace.

On the other hand, let's look at the image space. The zero vector is in here, since $A\mathbf{0} = \mathbf{0}$. Now let $\mathbf{y}_1, \mathbf{y}_2 \in \operatorname{image}(A)$, so that there exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ such that $A\mathbf{x}_i = \mathbf{y}_i$ for $i = 1, 2$. Now

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = y_1 + y_2$$

showing that $\mathbf{y}_1 + \mathbf{y}_2$ is hit by the element $\mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^n$, so $\mathbf{y}_1 + \mathbf{y}_2 \in \operatorname{image}(A)$. Similarly, if $\alpha \in \mathbb{R}$ then

$$A(\alpha \mathbf{x}_1) = \alpha A\mathbf{x}_1 = \alpha \mathbf{y}_1$$

showing that $\alpha \mathbf{y}_1 \in \operatorname{image}(A)$ as well. We conclude that $\operatorname{image}(A)$ is a susbspace. $\square$

4.4.1 Span

Since subspaces are built from adding and scalar multiplying vectors, you can build them by starting with a collection of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, and taking all linear combinations of these. This is called the span.

Definition 4.32

If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ then we define their *span* as

$$\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = \{c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k : c_i \in \mathbb{R}, i = 1, \ldots, k\}.$$

Proposition 4.33

For any collection of vectors $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$, $\operatorname{span}(S)$ is a subspace of $\mathbb{R}^n$.

Proof. The zero vector is the trivial span, taken by setting all the $c_i = 0$:

$$\mathbf{0} = 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k.$$

Suppose that $\mathbf{u}, \mathbf{v} \in \operatorname{span}(S)$ and write these as

$$\mathbf{u} = u_1\mathbf{x}_1 + u_2\mathbf{x}_2 + \cdots + u_k\mathbf{x}_k, \quad \mathbf{v} = v_1\mathbf{x}_1 + v_2\mathbf{x}_2 + \cdots + v_k\mathbf{x}_k.$$

The sum satisfies

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{x}_1 + (u_2 + v_2)\mathbf{x}_2 + \cdots + (u_k + v_k)\mathbf{x}_k$$

and if $\alpha \in \mathbb{R}$ then

$$\alpha \mathbf{u} = (\alpha u_1)\mathbf{x}_1 + (\alpha u_2)\mathbf{x}_2 + \cdots + (\alpha u_k)\mathbf{x}_k,$$

so $\mathbf{u} + \mathbf{v} \in \operatorname{span}(S)$ and $\alpha \mathbf{u} \in \operatorname{span}(S)$. $\square$

Spanning sets also provide a convenient way of writing down subspace. For example, the subspace from Example 4.26 can be written as

$$V = \left\{(x,y,z) \in \mathbb{R}^3 : x = y\right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

or

$$\mathbb{R}^n = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

where $\mathbf{e}_i$ are the standard coordinate vectors. In general, if $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ then $\operatorname{span}\{\mathbf{u}\}$ is a line through the origin, and $\operatorname{span}\{\mathbf{u}, \mathbf{v}\}$ is a plane through the origin.

Example 4.34

Suppose that $V = \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$. Show that $V = \operatorname{span}\{2\mathbf{u}, \mathbf{u} - \mathbf{v}\}$.

Solution. We can write

$$\mathbf{u} = \frac{1}{2}(2\mathbf{u}), \qquad \mathbf{v} = \frac{1}{2}(2\mathbf{u}) - (\mathbf{u} - \mathbf{v}).$$

Thus let $\mathbf{x} \in \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$, so that we can write $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$. This in turn can be written with the other vectors as

$$\begin{aligned} \mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} &= c_1\left[\frac{1}{2}(2\mathbf{u})\right] + c_2\left[\frac{1}{2}(2\mathbf{u}) - (\mathbf{u} - \mathbf{v})\right] \\ &= \left[\frac{1}{2}c_1 + \frac{1}{2}c_2\right](2\mathbf{u}) - c_2(\mathbf{u} - \mathbf{v}) \end{aligned}$$

showing that $\mathbf{x} \in \operatorname{span}\{2\mathbf{u}, \mathbf{u} - \mathbf{v}\}$. Hence $V \subseteq \operatorname{span}\{2\mathbf{u}, \mathbf{u} - \mathbf{v}\}$. Conversely, suppose that $\mathbf{y} \in \operatorname{span} 2\mathbf{u}, \mathbf{u} - \mathbf{v}$ so that $\mathbf{y} = c_1(2\mathbf{u}) + c_2(\mathbf{u} - \mathbf{v})$. Expanding this out we get

$$\mathbf{y} = [2c_1 + c_2]\,\mathbf{u} - c_2\mathbf{v}$$

so that $\mathbf{y} \in \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$. This shown that $\operatorname{span}\{2\mathbf{u}, \mathbf{u} - \mathbf{v}\} \subseteq V$. Both inclusions give the equality, as required. ■

We can use spanning sets to describe the null space and image space of an $m \times n$ matrix A. Let $\{\mathbf{c}_1, \ldots \mathbf{c}_n\}$ be the columns of A, and $\{\mathbf{h}_1, \ldots, \mathbf{h}_k\}$ be the basic solutions to the homogeneous equation $A\mathbf{x} = 0$. The image and null spaces are then

$$\operatorname{null}(A) = \operatorname{span}\{\mathbf{h}_1, \ldots, \mathbf{h}_k\}, \quad \operatorname{image}(A) = \operatorname{span}\{\mathbf{c}_1, \cdots, \mathbf{c}_n\}.$$

Example 4.35

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}.$$

Determine whether $\mathbf{u} \in \operatorname{span}\{\mathbf{v}, \mathbf{w}\}$.

Solution. We're asking whether there exist constants c_1, c_2 such that $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{w}$, which is analogous to solving the linear system given by the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & 0 & -5 \end{array}\right].$$

Reducing this matrix to REF, we get

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & 0 & -5 \end{array}\right] \xrightarrow{R_3+R_1\to R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{array}\right] \xrightarrow{-2R_3+R_2\to R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{-2R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Hence the answer is yes, and in fact $5\mathbf{u} - 2\mathbf{v} = \mathbf{w}$, which is easily checked by computing these vectors. ∎

4.5 Linear Independence

Spanning sets can have redundancies built in. For example, suppose $\mathbf{u} \in \mathbb{R}^n$ is a non-zero vector. The following sets are equal

$$\operatorname{span}\{\mathbf{u}, 2\mathbf{u}, 3\mathbf{u}, 4\mathbf{u}, 5\mathbf{u}\} = \operatorname{span}\mathbf{u}:$$

the vectors $2\mathbf{u}, \ldots, 5\mathbf{u}$ do not add anything new to the span. A less trivial example might be something like $\operatorname{span}\mathbf{u}, \mathbf{v}, \mathbf{w}$, where

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since we can write

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

this third vector is already in the span of the first two, and again does not add anything new. Effectively, we want to discuss when a set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ contain no redundancies. This is called *linear independence*.

Definition 4.36

Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a collection of vectors in $\mathbb{R}^n$. We say that S is a *linearly independent set* if whenever

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = 0$$

then $c_1 = c_2 = \cdots = c_k = 0$. We say that S is linearly *dependent* otherwise.

The three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are not linearly independent, since $\mathbf{u} + (-1)\mathbf{v} + (-1)\mathbf{w} = 0$. The advantage of being linearly independent is the following:

Theorem 4.37

If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is linearly independent, then every vector $\mathbf{v}$ in span $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ has a unique representation as

$$\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k.$$

How do you check that a collection of vectors is linearly independent? Notice, that $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n = 0$ is a homogeneous system of linear equations. Hence we can construct the matrix A whose columns are the $\mathbf{x}_i$, and ask if the only solution to this system is the trivial solution $\mathbf{0}$.

Example 4.38

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine whether these vectors are linearly independent.

Solution. We set up the matrix corresponding to the system $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = 0$, which gives us

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 4 & 2 \end{bmatrix}.$$

This system will have a unique solution if it has rank 3, so we reduce this to REF to get

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 4 & 2 \end{bmatrix} \xrightarrow[\mathrm{R}_3+\mathrm{R}_1\to\mathrm{R}_1]{-2\mathrm{R}_2+\mathrm{R}_1\to\mathrm{R}_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 6 & 2 \end{bmatrix} \xrightarrow{2\mathrm{R}_3+\mathrm{R}_2\to\mathrm{R}_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

After normalizing rows 2 and 3 we see that the matrix has rank 3, and thus the only solution to the homogeneous system is $\mathbf{0}$. We conclude that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linearly independent. ■

In the special case of only two vectors, the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if $\mathbf{u} = c\mathbf{v}$ for some non-zero c; namely, $\mathbf{u}$ and $\mathbf{v}$ are parallel. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ then you can check linear independence by computing the cross product $\mathbf{u} \times \mathbf{v}$.

Recall that the matrix A is invertible if and only if the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution. This is the same condition for the columns of A to be linearly independent. This means that we can add the following to our invertibility conditions:

Theorem 4.39

If A is a square $n \times n$ matrix, the following are equivalent:

1. A is invertible,
2. The columns of A are linearly independent,
3. The columns of A span $\mathbb{R}^n$,
4. The rows of A are linearly independent,
5. The rows of A span $\mathbb{R}^n$.

Now we want to find the Goldilocks zone between being linearly independent and being a spanning set for a subspace U. The first fact is following, which says that a subspace spanned by m vectors cannot admit a set with more than m linearly independent vectors.

Theorem 4.40

If $U = \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent subset of U, then $k \leq \ell$.

The goal then is to write U as the span of a linearly independent set. This would be the "best" way of writing U as span, in the sense that no fewer vectors would work, and more vectors would fail to be linearly independent.

Definition 4.41

Let $U \subseteq \mathbb{R}^n$ be a subspace. We say that a set $B = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a *basis* for U if B is linearly independent, and $U = \operatorname{span}(B)$.

For example, the subspace S from Example 4.26 has a basis

$$S = \left\{(x, y, z) \in \mathbb{R}^3 : x = y\right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \tag{4.3}$$

If A is a matrix, then $\operatorname{null}(A)$ has a basis given by the basic solutions to the homogeneous linear system $A\mathbf{x} = 0$.

It's important to note that bases are not unique. For example, a basis for $\mathbb{R}^3$ is given by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, since they span $\mathbb{R}^3$ are are linearly independent. However, the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

from Example 4.38 also spans $\mathbb{R}^3$. Our only recourse is that, no matter what basis we choose, it must always have the exact same number of vectors.

Theorem 4.42

If $U \subseteq \mathbb{R}^n$ is a subspace with bases B_1 and B_2, then $|B_1| = |B_2|$.

This allows us to define the dimension of a subspace.

Definition 4.43

We define the *dimension* of a subspace $U \subseteq \mathbb{R}^n$ as the number of elements in a basis for U. We write this as $\dim(U)$.

Hence the dimension of $\mathbb{R}^n$ is n, while the dimension of the subspace S in (4.3) is 2. In general, lines are one dimensional, planes are two dimensional.

Example 4.44

Suppose that A is an $n \times n$ invertible matrix, and $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ are a basis for a k-dimensional subspace U. Show that $\{A\mathbf{x}_1, \ldots, A\mathbf{x}_k\}$ is linearly independent and hence spans a k-dimensional subspace as well.

Solution. We begin by assuming that

$$c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2 + \cdots + c_k A\mathbf{x}_k = \mathbf{0},$$

for which our goal is to show that $c_1 = c_2 = \cdots = c_k = 0$. We can rewrite this equation as

$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = \mathbf{0},$$

showing that $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k$ is a solution to this homogeneous system, or equivalently is in (A). Since A is invertible, the homogeneous system only has the trivial solution, showing that

$$c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k = \mathbf{0}.$$

Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a linearly independent set, it must be the case that $c_1 = c_2 = \cdots = c_k = 0$, which is what we wanted to show. ■

Theorem 4.45

If U is a subspace of $\mathbb{R}^n$ then

1. U has a basis and $\dim(U) \leq n$,
2. Any linearly independent set in U can be enlarged to a basis for U,
3. Any spanning set for U can be reduced to a basis of U.

This theorem has several useful corollaries:

Corollary 4.46

If $U \subseteq \mathbb{R}^n$ is a subspace with $\dim(U) = m$, then $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ is linearly independent if and only if $\operatorname{span}(\mathcal{B}) = U$.

Corollary 4.47

If U, V are subspaces of $\mathbb{R}$ with $U \subseteq V$, then

1. $\dim(U) \leq \dim(V)$,
2. If $\dim(U) = \dim(V)$ then $U = V$.

4.6 Rank of a Matrix

When we first defined rank, I mentioned that I was unhappy with how it was defined. For example, it is not clear that rank is intrinsic to a matrix, that it does not depend on how the row reduction was performed. In this section, we will see a better definition of rank.

Definition 4.48

Let A be an $m \times n$ matrix, with columns $\mathbf{c}_i, i = 1, \ldots, n$ and rows $\mathbf{r}_i, i = 1, \ldots, m$. We define the *column space* as $\operatorname{col}(A) = \operatorname{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$, and the *row space* $\operatorname{row}(A) = \operatorname{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$.

If you go back and look at the definition of the image space, you'll see that it coincides with the column space. Hence the row space and image space of a matrix are identical. We can use row reduction to determine bases for the column and row spaces.

Theorem 4.49

If A is an $m \times n$ matrix, and $\operatorname{REF}(A)$ is A in row-echelon form, then the rows of $\operatorname{REF}(A)$ with leading ones form a basis for $\operatorname{row}(A)$. The columns of A such that $\operatorname{REF}(A)$ has leading ones form a basis for $\operatorname{col}(A)$.

Example 4.50

Find a basis for the row and columns spaces of

$$A = \begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{bmatrix}.$$

Solution. We put this into row-echelon form, via the following steps:

$$A = \begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{bmatrix} \xrightarrow[\mathrm{R}_1+\mathrm{R}_4\to\mathrm{R}_4]{(-2)\mathrm{R}_1+\mathrm{R}_2\to\mathrm{R}_2} \begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 0 & 0 & -12 & 9 & 3 \\ 0 & 0 & -12 & 9 & 3 \\ 0 & 0 & 12 & -9 & -3 \end{bmatrix}$$
$$\xrightarrow[(-1)\mathrm{R}_2+\mathrm{R}_3\to\mathrm{R}_3]{\mathrm{R}_2+\mathrm{R}_4\to\mathrm{R}_4} \begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 0 & 0 & -12 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We know the second row should have a leading one, but we'll leave it be to avoid fractions. Now the row space is spanned by the first a second rows of the REF of A, so

$$\operatorname{row}(A) = \operatorname{span}\{(1, -1, 5, -2, 2), (0, 0, -12, 9, 3)\}.$$

The leading ones occur in columns 1 and 3, so we revisit those columns of the original matrix to get

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -12 \\ 7 \end{bmatrix} \right\}.$$

■

The lends itself to two potentially different questions: Given the span of a set of vectors $U = \operatorname{span}\{\mathbf{v}_0, \ldots, \mathbf{v}_k\}$, you could be asked to find a subset of the $\{\mathbf{v}_0, \ldots, \mathbf{v}_k\}$ which form a basis for U, or you could just be asked for a basis of U. This makes a difference as to how you set up the question.

Example 4.51

Suppose

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -1 \\ 1 \\ -5 \end{bmatrix},$$

and $U = \operatorname{span}\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$.

1. Find a basis for U.
2. Find a subset of $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ which forms a basis for U,

Solution. Obviously part (2) is more restrictive than part (1).

1. Let's set up the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & 0 & 1 & -1 \\ 5 & -1 & 1 & 5 \end{bmatrix}$$

so that $U = \operatorname{row}(A)$. Row reducing gives

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1/3 & -1/3 \\ 0 & 1 & 2/3 & 10/3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and hence $\begin{bmatrix} 1 & 0 & 1/3 & -1/3 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & 2/3 & 10/3 \end{bmatrix}^T$ form a basis for U.

2. On the other hand, none of the vectors we found above were elements of the original set $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$. To do this, let's set up the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & 5 \end{bmatrix}.$$

Putting this into RREF gives

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we infer that $\{\mathbf{v}_0, \mathbf{v}_1\}$ forms a basis for U. ■

Since the rows and columns of the REF of A have the same number of leadings ones, we know that $\dim(\operatorname{row}(A)) = \dim(\operatorname{col}(A))$, which leads us to the more intrinsic definition of rank.

Definition 4.52

If A is a matrix, its *rank* is

$$\operatorname{rank}(A) = \dim(\operatorname{row}(A)) = \dim(\operatorname{col}(A)).$$

In addition, since $\operatorname{row}(A) = \operatorname{col}(A^T)$, we know that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$. The rank of the matrix A from Example 4.50 is 2.

Theorem 4.53

If A is an $m \times n$ matrix, the following are equivalent:

1. $\operatorname{rank}(A) = n$,
2. The rows of A span $\mathbb{R}^n$,
3. The columns of A are linearly independent in $\mathbb{R}^m$,
4. The $n \times n$ matrix A^TA is invertible,
5. $CA = I_n$ for some $n \times m$ matrix C (A has a left-inverse),
6. If $A\mathbf{x} = 0$ for $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} = 0$.

There are several interesting points to be made. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, then part (5) says that T has a left-inverse. The existence of a left-inverse is equivalent to injectivity of T, which in itself is equivalent to part (6) (Prove it!). Moreover, the matrix C in part (5) is $C = (A^TA)^{-1}A^T$, which is called the *Moore-Penrose* inverse.

Dual to Theorem 4.53 we have the following:

Theorem 4.54

If A is an $m \times n$ matrix, the following are equivalent:

1. $\operatorname{rank}(A) = m$,
2. The columns of A span $\mathbb{R}^m$,
3. The rows of A are linearly independent in $\mathbb{R}^n$,
4. The $m \times m$ matrix AA^T is invertible,
5. $AC = I_m$ for some $n \times m$ matrix C (A has a right-inverse),
6. The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

Once again, this condition is equivalent to surjectivity of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

4.7 Orthonormal Bases

If $U \subseteq \mathbb{R}^n$ is a subspace, a basis $B = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for U is said to be an *orthogonal basis* if

$$\mathbf{x}_i \cdot \mathbf{x}_j = 0 \quad \text{for all } i \neq j;$$

that is, each of the vectors in the basis are pairwise orthogonal. It is said to be an *orthonormal basis* if it is an orthogonal basis, and in addition $\|\mathbf{x}_i\| = 1$ for every $i = 1, \ldots k$. For example, the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Our goal in this section is to construct orthonormal bases.

To do this, we recall some facts about the dot product. If $\mathbf{v} \in \mathbb{R}^n$, then $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$, and in general $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

A benefit of an orthogonal basis is that the triangle inequality becomes an equality:

Theorem 4.55: Pythagorean Theorem

If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is an orthogonal set, then

$$\|\mathbf{x}_1 + \cdots + \mathbf{x}_k\|^2 = \|\mathbf{x}_1\|^2 + \cdots + \|\mathbf{x}_k\|^2.$$

The norm can be used to define the distance between two vectors, via $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$. The distance satisfies the following properties: For any $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$

1. $d(\mathbf{v}, \mathbf{w}) \geq 0$,
2. $d(\mathbf{v}, \mathbf{w}) = 0$ if and only if $\mathbf{v} = \mathbf{w}$,
3. $d(\mathbf{v}, \mathbf{w}) = d(\mathbf{w}, \mathbf{v})$,
4. $d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$.

We will not use the distance function much, but it is an exceptionally important tool.

Theorem 4.56: Cauchy-Schwarz Inequality

For any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,
$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|,$$
with equality precisely when $\mathbf{v}$ and $\mathbf{w}$ are parallel.

Since scaling does not affect direction, note that we can always normalize each of the vectors in an orthogonal basis to get an orthonormal basis. For example, the vectors
$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
are orthogonal (check), but are not orthonormal since $\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{2}$. Instead, the vectors
$$\hat{\mathbf{u}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \hat{\mathbf{v}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
form an orthonormal basis.

Theorem 4.57

If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is an orthogonal set of non-zero vectors, they are linearly independent.

Proof. Suppose that $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$. Choose an arbitrary $\mathbf{x}_i$, and dot this against the linear combination above, to get
$$\begin{aligned} \mathbf{0} &= \mathbf{x}_i \cdot (c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k) \\ &= c_1\mathbf{x}_i \cdot \mathbf{x}_1 + \cdots + c_i\mathbf{x}_i \cdot \mathbf{x}_i + \cdots + c_k\mathbf{x}_i \cdot \mathbf{x}_k \\ &= c_1\|\mathbf{x}_i\|^2. \end{aligned}$$
Since $\mathbf{x}_i \neq 0$, we know $\|\mathbf{x}_i\| \neq 0$ which in turn shows that $c_i = 0$. Since this can be done for any c_i, they must all be 0, showing that $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ are linearly independent as required. $\square$

Imitating the above proof yields the following important fact:

Theorem 4.58

If $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ is an orthogonal basis for a subspace U of $\mathbb{R}^n$ and $\mathbf{x} \in U$, then we can write $\mathbf{x} = c_1\mathbf{f}_1 + \cdots c_k\mathbf{f}_k$ where
$$c_i = \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2}.$$
In particular, if this is an orthonormal basis, then $c_i = \mathbf{x} \cdot \mathbf{f}_i$.

Your book calls these "Fourier coefficients," and there's a tenuous sense in which this is true. Strictly speaking, the term Fourier coefficients are usually reserved for when we apply this process to the vector space of functions, decomposed into its harmonics.

Note that each component of the decomposition

$$c_i \mathbf{f}_i = \frac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2} \mathbf{f}_i$$

is precisely the projection of $\mathbf{x}$ onto the line spanned by $\mathbf{f}_i$, according to Definition 4.9. This agrees with our intuition regarding the standard basis $\{\mathbf{e}_i : i = 1, \ldots, n\}$.

Example 4.59

Consider the vectors given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Confirm that this is an orthogonal basis for $\mathbb{R}^4$ and write $\mathbf{x} = (1, 2, 3, 4)^T$ in this basis.

Solution. It's not too hard to verify that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$, so I leave it to you as an exercise. Since they are mutually orthogonal, they are linearly independent, and so must be a basis for $\mathbb{R}^4$.

Now we can compute the coefficients for $\mathbf{x}$ as follows:

$$\begin{aligned} c_1 &= \frac{\mathbf{x} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} = \frac{10}{4} = \frac{5}{2} \\ c_2 &= \frac{\mathbf{x} \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2} = \frac{-4}{4} = -1 \\ c_3 &= \frac{\mathbf{x} \cdot \mathbf{v}_3}{\|\mathbf{v}_3\|^2} = \frac{-1}{2} = -\frac{1}{2} \\ c_4 &= \frac{\mathbf{x} \cdot \mathbf{v}_3}{\|\mathbf{v}_4\|^2} = \frac{-1}{2} = -\frac{1}{2}. \end{aligned}$$

Hence

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

which does indeed work out. ■

4.7.1 Projections Onto Subspaces

Definition 4.60

If $U \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, the *orthogonal complement* of U is

$$U^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{x} = 0, \forall \mathbf{x} \in U\}.$$

It's not too hard to check that $\{0\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{0\}$. In fact, we've used the notion of orthogonal complements to define planes in $\mathbb{R}^3$. Namely, given a normal vector/line $\mathbf{n}$, we defined the plane through the origin as the collection of all $\mathbf{x}$ such that $\mathbf{n} \cdot \mathbf{x} = 0$; that is, the orthogonal complement of $\operatorname{span}\{\mathbf{n}\}$.

Proposition 4.61

The orthogonal complement of a subspace is a subspace.

Proof. Let $U \subseteq \mathbb{R}^n$ be a subspace. Certainly $\mathbf{0} \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in U$, so $\mathbf{0} \in U^\perp$. Now suppose $\mathbf{u}, \mathbf{v} \in U^\perp$ and $r \in \mathbb{R}$. By definition, we know that

$$\mathbf{u} \cdot \mathbf{x} = \mathbf{0}, \qquad \mathbf{v} \cdot \mathbf{x} = \mathbf{0}, \qquad \forall \mathbf{x} \in U$$

hence

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{x} = \mathbf{0} + \mathbf{0} = \mathbf{0}, \qquad \forall \mathbf{x} \in U$$

and

$$(r\mathbf{u}) \cdot \mathbf{x} = r(\mathbf{u} \cdot \mathbf{x}) = r\mathbf{0} = \mathbf{0}, \qquad \forall \mathbf{x} \in U$$

so $U^\perp$ is closed under scalar multiplication and addition, making it a subspace, as required. □

To compute orthogonal subspaces, we can use the following lemma:

Lemma 4.62

If $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathbb{R}^n$ and $U = \operatorname{span}(S)$, then $U^\perp = \{\mathbf{x} : \mathbf{x} \cdot \mathbf{x}_i = 0, 1 \leq i \leq n\}$; that is, $U^\perp$ is the set of all vectors which are orthogonal to each of the $\mathbf{x}_i$.

Proof. We need to show a double subset inclusion. Suppose that $\mathbf{b} \in U^\perp$, so that $\mathbf{b} \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in U$. Notably, each $\mathbf{x}_i \in U$, so $\mathbf{b} \cdot \mathbf{x}_i = 0$. This shows that $U^\perp \subseteq \{\mathbf{x} : \mathbf{x} \cdot \mathbf{x}_i = 0\}$. Conversely, fix a vector $\mathbf{b} \in \{\mathbf{x} : \mathbf{x} \cdot \mathbf{x}_i = 0\}$, so that $\mathbf{b} \cdot \mathbf{x}_i = 0$ for each $i \in \{1, \ldots, n\}$. Let $\mathbf{y} \in U$, and write $\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$. Dotting against $\mathbf{b}$ gives

$$\mathbf{b} \cdot \mathbf{y} = c_1(\mathbf{b} \cdot \mathbf{x}_1) + c_2(\mathbf{b} \cdot \mathbf{x}_2) + \cdots + c_k(\mathbf{b} \cdot \mathbf{x}_k) = 0,$$

showing that $\mathbf{b} \in U^\perp$. Both inclusions give equality. □

Example 4.63

Define $U = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$. Find $U^\perp$.

Solution. By Lemma 4.62, it suffices to find the collection of all vectors which are orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ be such a vector, so that

$$\mathbf{x} \cdot \mathbf{v}_1 = x_1 - 2x_2 + 3x_3 = 0 \quad \text{and} \quad \mathbf{x} \cdot \mathbf{v}_2 = -x_1 + x_2 + x_3 = 0.$$

Both equations must be true simultaneously, resulting in a homogeneous system of equations. Row reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \end{bmatrix} \quad \text{with solutions} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} t.$$

Thus $U^\perp$ is the line spanned by this vector. ■

Theorem 4.64

If $U \subseteq \mathbb{R}^n$ is a subspace with orthogonal basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$, and $\mathbf{x} \in \mathbb{R}^n$, then the *projection* of $\mathbf{x}$ onto U is

$$\operatorname{proj}_U(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_3 + \cdots + \frac{\mathbf{x} \cdot \mathbf{f}_k}{\|\mathbf{f}_k\|^2}\mathbf{f}_k.$$

If $U \subseteq \mathbb{R}^n$ is a subspace, then just as in the one dimensional case we can write $\mathbf{x}$ as the sum of the its projection and an orthogonal component, given by $\mathbf{x}_0 = \mathbf{x} - \operatorname{proj}_U(\mathbf{x})$. This has several consequences:

1. Every vector $\mathbf{x} \in \mathbb{R}^n$ can be written as $\mathbf{x} = \mathbf{x}_U + \mathbf{x}_{U^\perp}$, with $\mathbf{x}_U \in U$ and $\mathbf{x}_{U^\perp} \in U^\perp$. This is written as $\mathbb{R}^n = U \oplus U^\perp$, though you'll have to wait until the following course to have a good understanding as to what this means.
2. $\dim(U^\perp) = n - \dim(U)$. This is known as the *codimension* of U; namely, $\operatorname{codim}(U) = \dim(U^\perp)$.
3. If $\mathbf{p} \in \mathbb{R}^n$, the point on U which minimizes the distance to $\mathbf{p}$ is $\operatorname{proj}_U(\mathbf{p})$.

Example 4.65

Find the projection of $\mathbf{b}$ onto the space U spanned by the orthogonal set $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, where

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{f}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 0 \\ -3 \end{bmatrix}.$$

Solution. We'll compute the projection onto each $\mathbf{f}_i$ separately, then add them together:

$$\frac{\mathbf{b} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{b} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{b} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = -\frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} + -\frac{7}{6}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ 4 \\ 1 \\ -5 \end{bmatrix}.$$ ■

Theorem 4.66

Let $U \subseteq \mathbb{R}^n$ be a subspace, and define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\mathbf{x}) = \operatorname{proj}_U(\mathbf{x})$. Then

1. T is a linear map,
2. $\operatorname{image}(T) = U$
3. $\operatorname{null}(T) = U^\perp$,
4. $T \circ T = T$.

The fact that $T_U(\mathbf{x}) = \operatorname{proj}_U(\mathbf{x})$ is a linear operator means it can actually be written as a matrix P. Property (4) additionally says that $P^2 = P$.

4.7.2 Gram-Schmidt Orthogonalization

Given an arbitrary basis, an algorithm for turning that basis into an orthogonal basis is given by the Gram-Schmidt algorithm. The idea is as follows: Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors. We can write $\mathbf{v}$ and as a linear combination $\mathbf{v} = \mathbf{u}_p + \mathbf{u}_o$ where $\mathbf{u}_p$ is parallel to $\mathbf{u}$ and $\mathbf{u}_o$ is *orthogonal* to $\mathbf{u}$. The orthogonal component is thus $\mathbf{u}_o = \mathbf{v} - \mathbf{u}_p$ where

$$\mathbf{u}_p = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}.$$

Taking $\{\mathbf{v}, \mathbf{u}_o\}$, we have an orthogonal set. Given a third vector $\mathbf{w}$, we can find the orthogonal component of the projection of $\mathbf{w}$ onto the plane spanned by $\{\mathbf{v}, \mathbf{u}_o\}$, which will give us an larger orthogonal set. We will iterate this process, and in doing so generate an orthogonal basis.

Gram-Schmidt:
Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for the subspace U, and define

$$\begin{aligned}
\mathbf{f}_1 &= \mathbf{x}_1 \\
\mathbf{f}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 \\
\mathbf{f}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 \\
&\vdots \\
\mathbf{f}_n &= \mathbf{x}_n - \frac{\mathbf{x}_n \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_n \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 - \cdots - \frac{\mathbf{x}_n \cdot \mathbf{f}_{n-1}}{\|\mathbf{f}_{n-1}\|^2}\mathbf{f}_{n-1}.
\end{aligned}$$

The set $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is an orthogonal basis for U.

Example 4.67

Consider the basis

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

of $\mathbb{R}^3$. Use Gram-Schmidt to turn this into an orthogonal basis.

Solution. We set $\mathbf{f}_1 = \mathbf{x}_1$, in which case

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix} \\
\mathbf{f}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{2}{11/2}\begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{2}{11}\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/11 \\ -15/11 \\ 5/11 \end{bmatrix}.
\end{aligned}$$

By multiplying to get rid of the fractions, our orthogonal set is

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \mathbf{f}_3 = \begin{bmatrix} 5 \\ -15 \\ 5 \end{bmatrix}$$

and you could normalize to turn this into an orthonormal basis, if desired. ■

Example 4.68

Using the orthogonal set $\{\mathbf{f}_1, \mathbf{f}_2\}$ you found in Example 4.67, let $U = \operatorname{span} \mathbf{f}_1, \mathbf{f}_2$. Determine the matrix form of the projection operator $\operatorname{proj}_U(\mathbf{x})$, and use this to find $\operatorname{proj}_U(\mathbf{b})$, where $\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

Solution. To find the matrix form for the projection operator, we input the standard basis vectors $\mathbf{e}_i$ for $i = 1, 2, 3$. Doing this, we get

$$\begin{aligned}
\operatorname{proj}_U(\mathbf{e}_1) &= \frac{1}{2}\mathbf{f}_1 + \frac{3}{22}\mathbf{f}_2 = \frac{1}{11}\begin{bmatrix} 10 & 3 & -1 \end{bmatrix}^T \\
\operatorname{proj}_U(\mathbf{e}_2) &= 0\mathbf{f}_1 + \frac{1}{11}\mathbf{f}_2 = \frac{1}{11}\begin{bmatrix} 3 & 2 & 3 \end{bmatrix}^T \\
\operatorname{proj}_U(\mathbf{e}_3) &= -\frac{1}{2}\mathbf{f}_1 + \frac{3}{11}\mathbf{f}_2 = \frac{1}{11}\begin{bmatrix} -1 & 3 & 10 \end{bmatrix}^T,
\end{aligned}$$

thus

$$A = \frac{1}{11}\begin{bmatrix} 10 & 3 & -1 \\ 3 & 2 & 3 \\ -1 & 3 & 10 \end{bmatrix}.$$

The projection is now determined by straightforward multiplication:

$$\operatorname{proj}_U(\mathbf{b}) = \begin{bmatrix} 10 & 3 & -1 \\ 3 & 2 & 3 \\ -1 & 3 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 12 \end{bmatrix}. \qquad \blacksquare$$

4.8 Approximating Solutions

In the wild, data is often imprecise and subject to small error bounds, making it impossible to solve the system $A\mathbf{x} = \mathbf{b}$ with the data at hand. Nonetheless, if you know that a theoretical answer should exist, you would strive to find the "closest possible" solution, attributing the lack of an exact answer to instrument precision.

If A is an $m \times n$ matrix, recall that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \operatorname{image}(A)$. Since $\operatorname{image}(A)$ is a subspace of $\mathbb{R}^m$, we can define a linear projection operator $T_{\operatorname{image}(A)} : \mathbb{R}^m \to \mathbb{R}^m$. The point in $\operatorname{image}(A)$ which is closest to $\mathbf{b}$ is then $T_{\operatorname{image}(A)}(\mathbf{b})$, and we are guaranteed that $A\mathbf{x} = T_{\operatorname{image}(A)}(\mathbf{b})$ has a solution.

Determining $T_{\operatorname{image}(A)}(\mathbf{b})$ is quite a chore: We would first find a basis for $\operatorname{image}(A)$ by putting A into row-echelon form. We would then apply Gram-Schmidt to orthogonalize that basis. Finally, we could project $\mathbf{b}$ onto the basis using the Fourier coefficients. Instead, let's use some algebraic trickery to arrive at the solution. Suppose that $\mathbf{z}$ is a solution to $A\mathbf{z} = T_{\operatorname{image}(A)}(\mathbf{b})$, so that $\mathbf{b} - A\mathbf{z}$ is orthogonal to $\operatorname{image}(A)$ ($A\mathbf{z} \in \operatorname{image}(A)$ and is the projection of $\mathbf{b}$). Hence for any $\mathbf{x} in \mathbb{R}^n$:

$$\begin{aligned} 0 = (A\mathbf{x}) \cdot (\mathbf{b} - A\mathbf{z}) &= (A\mathbf{x})^T(\mathbf{b} - A\mathbf{z}) \\ &= \mathbf{x}^T A^T \mathbf{b} - \mathbf{x}^T A^T A\mathbf{z} \\ &= \mathbf{x} \cdot (A^T\mathbf{b} - A^T A\mathbf{z}). \end{aligned}$$

Since this holds for all $\mathbf{x} \in \mathbb{R}^n$, it must be the case that $A^T A\mathbf{z} = A^T\mathbf{b}$. This is the *normal equation* for $\mathbf{z}$.

Note that if A is invertible, then

$$\mathbf{z} = (A^T A)^{-1} A^T \mathbf{b} = A^{-1}\mathbf{b},$$

which is a regular solution. If A is not even square though, the matrix $(A^T A)^{-1} A^T$ plays the role of the inverse, and is called the *Penrose inverse*. Additionally, the matrix $A^T A$ is symmetric, meaning that $(A^T A)^T = A^T A$, which means that you can save some time computing the elements of $A^T A$.

Example 4.69

Set up the best approximate solution to the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 6 \\ 3 & 2 & -1 \\ -1 & 4 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 6 \\ 0 \end{bmatrix}.$$

Solution. We must first compute the normal equation. Since A is 4×3, $A^T A$ will be 3×3, and we get

$$A^T A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 2 & 4 \\ -1 & 6 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 6 \\ 3 & 2 & -1 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 1 & 7 \\ 1 & 22 & -5 \\ 7 & -5 & 39 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 6 \\ 3 & 2 & -1 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & -1 & 2 & 4 \\ 1 & 6 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 16 \\ -5 \end{bmatrix}$$

The augmented matrix for $A^T A\mathbf{z} = A^T \mathbf{b}$ is

$$\left[\begin{array}{ccc|c} 15 & 1 & 7 & 25 \\ 1 & 22 & -5 & 16 \\ 7 & -5 & 39 & -5 \end{array}\right].$$

This is messy to solve, but can be done using a computer to get

$$\mathbf{z} \approx \begin{bmatrix} 1.87 \\ 0.56 \\ -0.38 \end{bmatrix}. \qquad \blacksquare$$

Another technique is curve fitting, used often in statistical analysis. The idea is that, give a bunch of data points, you want to find the line which best represents the data. So suppose your data consists of n-data points (x_i, y_i) for $i = 1, \ldots, n$, and you have a straight line $y = f(x) = mx+b$. Let d_i be the distance from your observed data (x_i, y_i) and the line $(x_i, f(x_i))$; namely, $d_i = |y_i - f(x_i)|$. We want to minimize the overall distances between the data points. However, since we often use calculus to assist us, and the absolute value function is not differentiable, we often minimize the sum of the distances squared $d_1^2 + d_2^2 + \cdots + d_n^2$. Since the square function is monotonic for positive inputs, and is differentiable, this achieves the same affect but behaves nicer with derivatives. Hence we want to minimize

$$\|\mathbf{y} - \mathbf{f}\|^2$$

where $\mathbf{y} = (y_1, \ldots, y_n)^T$ and $\mathbf{f} = (f(x_1), \ldots, f(x_n))^T = (mx_1 + b, \ldots, mx_n + b)^T$. If we set

$$M = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

then we're trying to find the best approximation to the system $M\mathbf{x} = \mathbf{y}$.

Least Square Approximation: Given data points $(x_i, y_i), i = 1, \ldots, n$, let $M, \mathbf{x}, \mathbf{y}$ be as above. The Least Square Approximation is the line given by $y = mx + b$, where $\mathbf{x} = (m, b)^T$ satisfies the normal equations $M^T M\mathbf{z} = M^T \mathbf{y}$.

If instead of using lines you wish to use higher order polynomials, you can change the matrix M and $\mathbf{x}$ to include higher order terms

$$M = \begin{bmatrix} x_1^n & \cdots & x_1^2 & x_1 & 1 \\ x_2^n & \cdots & x_2^2 & x_2 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ x_k^n & \cdots & x_k^2 & x_k & 1 \end{bmatrix}$$

known as a *Vandermonde* matrix.

Example 4.70

Find the least squares approximation to the data $(0,1),(1,3),(2,2)$.

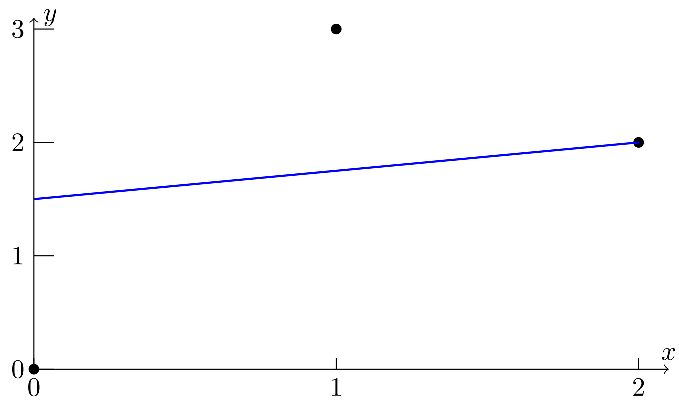

Figure 15: The least squares solution to Example 4.70

Solution. Our matrices are

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

with the normal equation components

$$M^T M = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}, \qquad M^T \mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}.$$

Our usual techniques show that $M^T M\mathbf{z} = M^T\mathbf{y}$ has solution $\mathbf{z} = (1/2, 3/2)^T$, so the least squares approximation is the line $y = (x+3)/2$. ■