Spectral Analysis (Theory)

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Idea: Decompose a **stationary** time series $\{X_t\}$ into a combination of "sinusoids", with random (and uncorrelated) coefficients.

Example: A periodic time series

Consider

$$X_t = Asin(2\pi wt) + Bcos(2\pi wt)$$

where A, B are uncorrelated random variables with mean zero and variance one.

Example: A periodic time series

i) Find its mean.
ii) Find its autocovariance function.
iii) Find its autocorrelation function.

Solution i)

i) $E(X_t) = E[Asin(2\pi wt) + Bcos(2\pi wt)]$
$= sin(2\pi wt)E(A) + cos(2\pi wt)E(B) = 0$

Solution ii)

Let us recall a result that will allow us to find the autocovariance function, more easily. Let X and Y be to independent random variables. Now we define $U_1 = aX + bY$ and $U_2 = cX + dY$, where a, b, c, and d are constants.

$$cov(U_1, U_2) = cov[aX + bY, cX + dY]$$
$$= ac\ cov(X, X) + ad\ cov(X, Y) + bc\ cov(Y, X) + bd\ cov(Y, Y)$$

(since X and Y are independent, $cov(X, Y) = 0$)

$$cov(U_1, U_2) = ac\ cov(X, X) + bd\ cov(Y, Y) = ac\ Var(X) + bd\ Var(Y).$$

Solution ii)

Letting $X = A$, $Y = B$, $a = cos(2\pi wt)$, $b = sin(2\pi wt)$, $c = cos[2\pi w(t-k)]$, $d = sin[2\pi w(t-k)]$, we have that
$cov(X_t, X_{t-k}) = cos(2\pi wt)cos[2\pi w(t-k)] + sin(2\pi wt)sin[2\pi w(t-k)]$
$cov(X_t, X_{t-k}) = cos\{2\pi[wt-(wt-wk)]\} = cos[2\pi(wt-wt+wk)] = cos[2\pi wk]$.
Thus,
$\gamma(k) = cos[2\pi wk]$.

Solution iii)

Since $\gamma(0) = cos(0) = 1$, we have that
$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = cos[2\pi wk]$.

Homework?

If $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary sequences, i.e., if X_r and Y_s are uncorrelated for every r and s, show that $\{X_t + Y_t\}$ is stationary with autocovariance function equal to the sum of the autocovariance functions of $\{X_t\}$ and $\{Y_t\}$.

The autocovariance of the sum of two uncorrelated time series is the sum of their autocovariances. Thus, the autocovariance of a sum of random sinusoids is a sum of sinusoids with the corresponding frequencies:
$X_t = \sum_{j=1}^{m}[A_j sin(2\pi w_j t) + B_j cos(2\pi w_j t)]$
$\gamma(h) = \sum_{j=1}^{m} \sigma_j^2 cos(2\pi w_j h)$,
where A_j, B_j are uncorrelated, mean zero, and $Var(A_j) = Var(B_j) = \sigma_j^2$.

Spectrum of a stationary random process

Consider a stationary random sequence $\{Y_t\}$ with autocovariance function $\gamma(k) = cov(Y_t, Y_{t-k})$. The corresponding autocovariance generating function is the function

$$G(z) = \sum_{k=-\infty}^{\infty} \gamma(k) z^k,$$

whose argument, z, is a complex variable.

Spectrum of a stationary random process

If we choose $z = e^{-i(2\pi w)}$ where w is a real variable, we obtain the **spectrum** of $\{Y_t\}$ (a.k.a. **spectral density**)

$$f(w) = G(e^{-i(2\pi w)}) = \sum_{k=-\infty}^{\infty} \gamma(k) e^{-2\pi w_i k}$$

Spectrum of a stationary random process

Because $\gamma(k) = \gamma(-k)$ and $e^{iw_*} + e^{-iw_*} = 2cos(w_*)$ (we will justify this later) we can also write $f(w)$ as

$$f(w) = \gamma(0) + \sum_{k=1}^{\infty} \gamma(k) e^{-2\pi w_i k} + \sum_{k=1}^{\infty} \gamma(k) e^{2\pi w_i k}$$

or

$$f(w) = \gamma(0) + 2 \sum_{k=1}^{\infty} \gamma(k) cos(2\pi kw)$$

Spectrum of a stationary random process

If σ^2 denotes the variance of Y_t we can similarly define a **normalized spectrum** (a.k.a. **normalized spectral density function**),

$$f^*(w) = \sum_{k=-\infty}^{\infty} \rho(k) e^{-2\pi w_i k}$$

or

$$f^*(w) = \frac{f(w)}{\sigma^2} = 1 + 2\sum_{k=1}^{\infty} \rho(k) cos(2\pi kw)$$

Example: White noise

Let Y_t be the process defined by $Y_t = W_t$ where W_t is white noise with mean 0 and variance σ_w^2. Compute the spectral density and normalized spectral density of Y_t.

Solution

First, we check that Y_t is stationary.
$E(Y_t) = 0$
Let $t \neq s$
$cov(Y_t, Y_s) = E[Y_t Y_s] = E[W_t W_s]$ (since W_s and W_t are independent)
$cov(Y_t, Y_s) = (0)(0) = 0.$
If $t = s$
$cov(Y_t, Y_t) = E[Y_t^2] = E[W_t^2] = \sigma_w^2$

Solution

$$\gamma(k) = \begin{cases} \sigma_w^2, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Solution

We know that the spectral density of Y_t is given by

$$f(w) = \sum_{k=-\infty}^{\infty} \gamma(k) e^{-2\pi w_i k}.$$

In this case,

$$f(w) = \gamma(0)e^{-0} + 0 = \gamma(0) = \sigma_w^2.$$

Solution

Then, we have that the normalized spectral density of Y_t is

$$f^*(w) = \sum_{k=-\infty}^{\infty} \rho(k) e^{-2\pi w_i k} = \rho(0) e^{-0} + 0 = \rho(0) = 1.$$

This is the origin of the name white noise: it is like white light, which is a uniform mixture of all frequencies in the visible spectrum.

Example. AR(1)

Let Y_t be the process defined by $Y_t = \phi Y_{t-1} + W_t$ (where W_t $|\phi| < 1$ is white noise with mean 0 and variance σ_W^2). Compute the spectral density and normalized spectral density of Y_t.

Solution

Since we have an AR(1) process with $|\phi| < 1$, we know it is stationary. It can be shown that

$\gamma(0) = \frac{\sigma_w^2}{1-\phi^2}$

$\gamma(1) = \frac{\phi\sigma_w^2}{1-\phi^2}$

. . .

$\gamma(k) = \frac{\phi^k\sigma_w^2}{1-\phi^2}$

(actually, we have done it a couple of times together, in class)

Solution

Thus

$$f(w) = \sum_{k=-\infty}^{\infty} \gamma(k) e^{-2\pi w_i k}$$
$$= \frac{\sigma_w^2}{1-\phi^2}\left[1 + \sum_{k=1}^{\infty} \phi^k e^{-2\pi iwk} + \sum_{k=1}^{\infty} \phi^k e^{2\pi iwk}\right]$$
$$= \frac{\sigma_w^2}{1-\phi^2}\left[1 + \sum_{k=1}^{\infty}(\phi e^{-2\pi iw})^k + \sum_{k=1}^{\infty}(\phi e^{2\pi iw})^k\right]$$
$$= \frac{\sigma_w^2}{1-\phi^2}\left[1 + \frac{\phi e^{-2\pi iw}}{1-\phi e^{-2\pi iw}} + \frac{\phi e^{2\pi iw}}{1-\phi e^{2\pi iw}}\right]$$

Solution

(After a little bit of algebra... Remember? We did it together)

$$= \frac{\sigma_w^2}{1-\phi^2}\left[\frac{1-\phi^2}{(1-\phi e^{-2\pi i w})(1-\phi e^{2\pi i w})}\right]$$
$$= \frac{\sigma_w^2}{1-\phi[e^{2\pi i w}+e^{-2\pi i w}]+\phi^2}$$

(next, I am going to remind you something that you know from Calc, that will allow us to simplify the last expression)

Solution

Recalling that

$sin(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}$

$cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k}}{(2k)!}$ and

$exp(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{k!}$

Solution

$$
\begin{aligned}
exp(ix) &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \\
&= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right] \\
&+ i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right]
\end{aligned}
$$

Thus,
$exp(ix) = cos(x) + i\ sin(x)$
Euler's Formula.

Solution(spectral density)

Using Euler's Formula
$exp(2\pi iw) = cos(2\pi w) + i\ sin(2\pi w)$
$exp(-2\pi iw) = cos(2\pi w) - i\ sin(2\pi w)$.
So, $exp(2\pi iw) + exp(-2\pi iw) = 2cos(2\pi w)$
Finally!!,

$$f(w) = \frac{\sigma_w^2}{1 - 2\phi cos(2\pi w) + \phi^2}$$

Solution(normalized spectral density)

We know that

$$f^*(w) = \sum_{k=-\infty}^{\infty} \rho(k) e^{-2\pi w_i k}.$$

So ... Homework? (It should be easy, Right? Just use $\rho(k)$ instead of $\gamma(k)$).

Homework Problem

Let Y_t be the process defined by

$$Y_t = W_t + \beta W_{t-1}$$

(where $W_t \sim WN(0, \sigma_W^2)$). Compute the spectral density and normalized spectral density of Y_t.