A first-order moving-average process, written as MA(1), has the general equation

\[ x_t = w_t + bw_{t-1} \]

where \( w_t \) is a white-noise series distributed with constant variance \( \sigma_w^2 \).
We must compute $\gamma(k)$, which is defined as the autocovariance of the process at lag $k$. For simplicity, assume that the mean has been subtracted from our data, so that $x_t$ has zero mean. Then

$$\gamma(k) = E(x_t x_{t-k})$$
The Autocovariance for MA(1) Models

\[ \gamma(k) = E[(w_t + bw_{t-1})(w_{t-k} + bw_{t-k-1})] \]
\[ = E(w_tw_{t-k} + bw_tw_{t-k-1} + bw_{t-1}w_{t-k} + b^2w_{t-1}w_{t-k-1}) \]
\[ = E(w_tw_{t-k}) + E(bw_tw_{t-k-1}) + E(bw_{t-1}w_{t-k}) + E(b^2w_{t-1}w_{t-k-1}) \]
Now set $k = 0$ and recall that $\gamma(0) = \sigma^2_{MA}$, the variance of your series.

$$\gamma(0) = \sigma^2_{MA} = E(w_t^2) + bE(w_t w_{t-1}) + bE(w_{t-1} w_t) + b^2 E(w_{t-1}^2)$$

$$\gamma(0) = \sigma^2_{MA} = \sigma^2_w + 0 + 0 + b^2 \sigma^2_w = (1 + b^2)\sigma^2_w.$$
Now set \( k = 1 \).

\[
\gamma(1) = \mathbb{E}(w_t w_{t-1}) + b \mathbb{E}(w_t w_{t-2}) + b \mathbb{E}(w_{t-1}^2 w_{t-1}) + b^2 \mathbb{E}(w_{t-1} w_{t-2})
\]

\[
\gamma(1) = b \sigma_w^2.
\]
For $k > 1$, we will obtain $\gamma(k) = 0$, since

$E[(w_t + bw_{t-1})(w_{t-k} + bw_{t-k-1})]$ will contain only terms whose expected value is zero.

**Note.** For an MA(1), the autocovariance function truncates (i.e., it is zero) after lag 1.
The Autocorrelation for MA(1) Models

\[ \rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1. \]

\[ \rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{b}{1 + b^2}. \]

\[ \rho(k) = 0 \text{ for all } k > 1. \]
For the qth-order MA process, we can use a similar derivation to show that the autocovariance function, $\gamma(k)$, truncates after lag q. Once again

$$\gamma(k) = E(x_t x_{t-k})$$
For $k = 0$, we obtain

$$\gamma(0) = \sigma^2_{MA} = (b_0^2 + b_1^2 + b_2^2 + \ldots + b_q^2)\sigma^2_w.$$ 

For $k = 1$, we obtain

$$\gamma(1) = (b_1 b_0 + b_2 b_1 + \ldots + b_q b_{q-1})\sigma^2_w.$$
In general, we obtain the basic equation

\[ \gamma(k) = \sigma_w^2 \sum_{s=0}^{q} b_s b_{s-k}. \]
Consider the MA(2) process, which is given by

\[ x_t = w_t + b_1 w_{t-1} + b_2 w_{t-2}, \]

where \( w_t \) is again a white-noise process.
At this point, it should be easy to see that

\[
\gamma(0) = \sigma_{MA}^2 = (1 + b_1^2 + b_2^2)\sigma_w^2
\]
\[
\gamma(1) = (b_1 + b_1 b_2)\sigma_w^2
\]
\[
\gamma(2) = b_2\sigma_w^2
\]
\[
\gamma(k) = 0 \text{ for } k > 2.
\]
\( \rho(0) = 1 \)
\( \rho(1) = \frac{b_1 + b_1 b_2}{1 + b_1^2 + b_2^2} \)
\( \rho(2) = \frac{b_2}{1 + b_1^2 + b_2^2} \)
\( \rho(k) = 0 \) for \( k > 2 \).

Thus, we see that the autocorrelation function for an MA(2) process truncates after two lags.
Suppose that we have an MA(1) model

\[ x_t = w_t + bw_{t-1}. \]

Then,

\[ x_{t-1} = w_{t-1} + bw_{t-2}. \]

Solve this equation for \( w_{t-1} \) and substitute the result back into \( x_t = w_t + bw_{t-1} \).
MA(1) is an AR(∞)

This gives

\[ x_t = w_t + b(x_{t-1} - bw_{t-2}) \]
\[ = bx_{t-1} + w_t - b^2w_{t-2} \]

(Now, we repeat the process with \( w_{t-2} \))
$x_{t-2} = w_{t-2} + bw_{t-3}$.

Solve this equation for $w_{t-2}$ and substitute the result back into $x_t = bx_{t-1} + w_t - b^2w_{t-2}$.

$$x_t = bx_{t-1} - b^2x_{t-2} + w_t + b^3w_{t-3}$$
MA(1) is an AR(∞)

We can continue indefinitely as long as $b^s$ goes to zero (i.e., $|b| < 1$) to obtain

$$x_t = w_t + bx_{t-1} - b^2x_{t-2} + b^3x_{t-3} - \ldots + \ldots$$

This is an AR(∞) process, but it only holds under the invertibility condition that $|b| < 1$. 
Consider the following first-order MA processes:

A: $x_t = w_t + \theta w_{t-1}$

B: $x_t = w_t + \frac{1}{\theta} w_{t-1}$
It can easily be shown that these two different processes have exactly the same autocorrelation function (Right?)

\[
\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1.
\]

\[
\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{1 + \theta^2}.
\]

\[
\rho(k) = 0 \text{ for all } k > 1.
\]
If $|\theta| < 1$, the series $(AR(\infty))$ for A converges whereas that for B does not. Thus if $|\theta| < 1$, model A is said to be invertible whereas model B is not. The imposition of the invertibility condition ensures that there is a unique MA process for a given autocorrelation function.
Simulated Examples of the MA(1) Model

\[ x_t = \omega_t + b_1 \omega_{t-1} \]

There are two cases, positive and negative values.
Case i) \( b_1 = -0.7 \)
Case ii) \( b_1 = 0.3. \)
R Code

```r
set.seed(9999);

# simulating MA(1);
ma1.sim<-arima.sim(list(ma = c(-0.7)),
n = 100, sd=2);

plot.ts(ma1.sim, ylim=c(-6,8),main="MA(1), b = -0.7, n=100");
```
Scatterplot

MA(1), b = -0.7, n=100

Time
ma1.sim
0 20 40 60 80 100
-6 0 4 8

The Moving Average Models MA(1) and MA(2)
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Autocorrelation Function

```r
acf(ma1.sim);
```
Autocorrelation Function, case i)

Series ma1.sim

Lag
ACF
0 5 10 15 20
−0.4 0.2 0.8

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set.seed(9999);

# simulating MA(1);
ma1.sim<-arima.sim(list(ma = c(0.3)),
n = 100, sd=2);

plot.ts(ma1.sim, ylim=c(-6,8),main="MA(1), b = 0.3, n=100");
MA(1), $b = 0.3$, $n=100$
Autocorrelation Function, case ii)

```plaintext
acf(ma1.sim);
```
Autocorrelation Function, case ii)

Series ma1.sim

Lag
ACF
Series  ma1.sim

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The Moving Average Models MA(1) and MAj
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Simulated Examples of the MA(2) Model

\[ x_t = w_t + b_1 w_{t-1} + b_2 w_{t-2}. \]

Case i) \( b_1 = 1.50 \) and \( b_2 = -0.56 \)
Case ii) \( b_1 = 0.50 \) and \( b_2 = 0.24 \)
Case iii) \( b_1 = -0.5 \) and \( b_2 = 0.24 \)
Case iv) \( b_1 = 1.20 \) and \( b_2 = -0.72 \)
b1<- 1.5;
b2<- -0.56;

set.seed(9999);

# simulating MA(2);
ma2.sim<-arima.sim(list(ma = c(b1,b2)),
n = 100, sd=2);

plot.ts(ma2.sim, ylim=c(-8,10),main="MA(2), case i");
MA(2), case i)

Time
ma2.sim

-5 0 5 10

0 20 40 60 80 100

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acf(ma2.sim);
Autocorrelation Function, case i)

Series ma2.sim

Lag

ACF

0 5 10 15 20

-0.2 0.4 1.0

Lag
b1<- 0.5;
b2<- 0.24;

set.seed(9999);

# simulating MA(2);
ma2.sim<-arima.sim(list(ma = c(b1,b2)),
n = 100, sd=2);

plot.ts(ma2.sim, ylim=c(-8,10),main="MA(2), case ii");
Scatterplot

MA(2), case ii)
Autocorrelation Function, case ii)

```r
acf(ma2.sim);
```
Series ma2.sim

Autocorrelation Function, case ii)
b1 <- -0.5;

b2 <- 0.24;

set.seed(9999);

# simulating MA(2);
ma2.sim <- arima.sim(list(ma = c(b1, b2)),
n = 100, sd=2);

plot.ts(ma2.sim, ylim=c(-8,10),main="MA(2), case ii");
MA(2), case iii)
Autocorrelation Function, case iii)

\texttt{acf(ma2.sim);}
Autocorrelation Function, case iii)

Series ma2.sim

Lag

ACF

Series ma2.sim

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The Moving Average Models MA(1) and MAj

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b1 <- 1.20;
b2 <- -0.72;

set.seed(9999);

# simulating MA(2);
ma2.sim <- arima.sim(list(ma = c(b1, b2)),
n = 100, sd = 2);

plot.ts(ma2.sim, ylim = c(-8, 10), main = "MA(2), case ii");
MA(2), case iv)
Autocorrelation Function, case iv)

```r
acf(ma2.sim);
```
Autocorrelation Function

Series ma2.sim

Lag

ACF

Series ma2.sim

Lag

ACF

Series ma2.sim

Lag

ACF

Series ma2.sim

Lag

ACF

Series ma2.sim

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Series ma2.sim

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Series ma2.sim

Lag

ACF