The Duality of MA and AR Processes

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Because ARMA processes involve linear combinations of the elements of random sequences $\{x_t\}$ and $\{w_t\}$, it is useful to have a shorthand notation for the operation of shifting the time index.
For this, we use the **backward shift operator** B, where Bx_t is to be interpreted as x_{t-1}. More usefully, powers of B are to be interpreted as successive applications of B; for example $B^2x_t = B(Bx_t) = Bx_{t-1} = x_{t-2}$. Finally, 1 indicates a null operation, $1x_t = x_t$, and mathematical functions of B are to be interpreted as their formal series expansions. For example,

$$\left(1 - \frac{B}{2}\right)^{-1} x_t = \left(\sum_{k=0}^{\infty} \left(\frac{B}{2}\right)^k\right) x_t = \sum_{k=0}^{\infty} 2^{-k} B^k x_t = \sum_{k=0}^{\infty} 2^{-k} x_{t-k}$$

The backward-shift operator, B, operates on a series to move it back one time unit, as follows:

t	x_t	Bx_t
1	2.1	·
2	5.6	2.1
3	0.3	5.6
4	10.0	0.3

The Backward-shift Operator

A new time series, Bx_t, is created by this transformation. In mathematical notation the backward shift is

$$Bx_t = x_{t-1}.$$

If we want to go back two time units, we apply B twice. In mathematical notation,

$$B(Bx_t) = B(x_{t-1}) = x_{t-2}$$

or

$$B^2 x_t = x_{t-2}.$$

In general, to shift back k time units,

$$B^k x_t = x_{t-k}.$$

Example, AR(2)

We can use powers of B to write AR models. For example, the AR(2) process,

$$\begin{aligned} x_t &= a_1 x_{t-1} + a_2 x_{t-2} + w_t \\ &= a_1 B(x_t) + a_2 B^2(x_t) + w_t \\ &= (a_1 B + a_2 B^2) x_t + w_t \end{aligned}$$

or

$$(1 - a_1 B - a_2 B^2) x_t = w_t.$$

Example, AR(p)

Another example, the AR(p) process,

$$
\begin{aligned}
x_t &= a_1 x_{t-1} + a_2 x_{t-2} + .. + a_p x_{t-p} + w_t \\
&= a_1 B(x_t) + a_2 B^2(x_t) + ... + a_p B^p(x_t) + w_t
\end{aligned}
$$

$$(1 - a_1 B - a_2 B^2 - ... - a_p B^p) x_t = w_t.$$

Example, MA(2)

The MA(2) model has a similar representation:

$$
\begin{aligned}
x_t &= b_1 w_{t-1} + b_2 w_{t-2} + w_t \\
&= w_t + b_1 B(w_t) + b_2 B(w_t)
\end{aligned}
$$

$$x_t = (1 + b_1 B + b_2 B^2) w_t$$

Example, MA(q)

The MA(q) model can be represented as follows:

$$x_t = b_1 w_{t-1} + b_2 w_{t-2} + w_t + ... + b_q w_{t-q}$$
$$= w_t + b_1 B(w_t) + b_2 B(w_t) + ... + b_q B^q(w_t)$$
$$x_t = (1 + b_1 B + b_2 B^2 + ... + B^q) w_t$$

The AR/MA duality revisited

We will show that the proces

$$x_t = ax_{t-1} + w_t$$

can be rewritten as an infinite moving average.

The AR/MA duality revisited

Using the new notation, consider the moving average

$$x_t^* = w_t + aw_{t-1} + a^2 w_{t-2} + ...$$
$$= [1 + aB + (aB)^2 + ...]w_t$$
$$x_t^* = \left[\sum_{i=0}^{\infty}(aB)^i\right] w_t$$

(It is not so difficult to show that $\left[\sum_{i=0}^{\infty}(aB)^i\right] = \frac{1}{1-aB}$, provided that $|a| < 1$. Remember? We did it in class).

Therefore,

$x_t^* = \left(\frac{1}{1-aB}\right) w_t$ (provided that $|a| < 1$.

The AR/MA duality revisited

Our last equation can be written as

$(1 - aB)x_t^* = w_t$

$x_t^* - aBx_t^* = w_t$

$x_t^* - ax_{t-1}^* = w_t$

$x_t^* = ax_{t-1}^* + w_t$

Here we see that polynomial operators have **inverses** and that **infinite moving-average processes are finite autoregressive processes**.

The AR/MA duality revisited (another example)

Consider a first-order moving -average process

$x_t = w_t - b_1 w_{t-1}$

In operator notation,

$x_t = (1 - b_1 B) w_t$

or equivalently,

$\left(\frac{1}{1 - b_1 B}\right) x_t = w_t$

The AR/MA duality revisited (another example)

$$\left(1 + b_1 B + b_1^2 B^2 + ...\right) x_t = w_t$$
$$x_t + b_1 B x_t + b_1^2 B^2 x_t + ... = w_t$$
$$x_t + b_1 x_{t-1} + b_1^2 x_{t-2} + ... = w_t$$
$$x_t = -b_1 x_{t-1} - b_1^2 x_{t-2} - ... + w_t$$

The reverse is also true; **finite moving-average processes are infinite autoregressive processes**.

ARMA or "mixed" processes

Suppose that we have a set of data, x_t, that we fit an AR(1) and obtain a residual η_t

$$x_t = ax_{t-1} + \eta_t$$

Suppose further that the ACF of the residual had only one nonzero value, so that we suspected that the residual was an MA(1) process, with parameter b.

ARMA or "mixed" processes

Then we could write

$$x_t = ax_{t-1} + w_t + bw_{t-1}$$

where w_t is white noise Alternatively, we can write this as

$$x_t - ax_{t-1} = w_t + bw_{t-1}$$

or

$$(1 - aB)x_t = (1 + bB)w_t$$

This is called a mixed or autoregressive moving-average (ARMA) process.

ARMA or "mixed" processes

Because the MA(1) process can also be written as an infinite AR process (if $|b| < 1$), our last equation can be rewritten as follows:

$$(1 - aB)\left(\frac{1}{1 + bB}\right) x_t = w_t$$

or

$$(1 - aB)\left(1 - bB + b^2 B^2 - b^3 B^3 + ...\right) x_t = w_t$$

ARMA or "mixed" processes

$$\left(1 - (a+b)B + (ab+b^2)B^2 + ...\right) x_t = w_t$$

This is an infinite-order AR process. Eventually, the powers of a and b will become small, so it will be reasonable to approximate this process with a finite-order AR of reasonably low order.

These results are true more generally under some appropriate conditions. If we have two polynomials in B, MA(B) and AR(B), and an ARMA model

$$AR(B)x_t = MA(B)w_t$$

It is possible to write the model as an infinite AR process,

$$\left[\frac{AR(B)}{MA(B)}\right] x_t = w_t$$

or and infinite MA process,

$$x_t = \left[\frac{MA(B)}{AR(B)}\right] w_t$$

and approximate either by finite processes.

The condition necessary for "dividing" by AR(B) is that the AR process be stationary. However, since all finite MA processes are stationary, this condition will not be adequate for dividing by MA(B). The condition necessary for dividing by MA(B) is that the MA process be "invertible". Both conditions can be expressed in terms of roots of the polynomials AR(B) and MA(B).

Roots of polynomials: stationarity and invertibility

The polynomial $x^2 - x - 2$ can also be written $(x - 2)(x - 1)$, and, in general, any polynomial of order n can be rewritten as a product

$$(x - s_1)(x - s_2)...(x - s_n)$$

where the s_k are, in general, complex numbers. When x equals any of these numbers, the polynomial is zero, and the numbers s_k are called the roots of the polynomial.

Roots of polynomials: stationarity and invertibility

The polynomials AR(B) and MA(B) also have roots determined entirely by their parameters. In general, every root s_k of the generating polynomials AR(B) or MA(B) will be a complex number

$$s_k = u_k + iv_k$$

where u_k and v_k are real numbers and $i = \sqrt{-1}$.
The modulus of a complete number s_k is written $|s_k|$ and defined as

$$|s_k| = \sqrt{u_k^2 + v_k^2}$$

AR(1)...Again

In general, and AR process will be stationary if all roots of AR(B) have modulus greater than 1. For the AR(1) case, $\text{AR(B)} = 1 - aB$, and the root of the polynomial, $1 - az$ is $s_1 = \frac{1}{a}$.
The condition $|s_1| > 1$ is the same as $|a| < 1$.
(The condition we are familiar with for the AR(1) process to be stationary!)

We can divide by the AR(B) polynomial to get equation

$$x_t = \left[\frac{MA(B)}{AR(B)} \right] w_t$$

whenever the roots of $AR(B)$ have modulus greater than 1.

Similarly, we can divide by the MA(B) polynomial to get equation

$$\left[\frac{AR(B)}{MA(B)}\right] x_t = w_t$$

whenever the roots of the MA(B) polynomial have modulus greater than 1; in this case the MA(B) operator is called invertible, and the conditions on its roots are called invertibility conditions.

MA(1)...Again

The basic model for MA(1) is

$$x_t = w_t + b_1 w_{t-1} = (1 + b_1 B) w_t,$$

and the root of the polynomial, $1 + b_1 z$ is $s_1 = \frac{-1}{b_1}$. The condition $|s_1| > 1$ is the same as $|b_1| < 1$.

MA(1) is invertible when $|b_1| < 1$.

Example

Consider the MA(2) process

$$x_t = w_t - 0.1w_{t-1} + 0.21w_{t-2}$$

Is this model invertible? Why?