Characteristics of Time Series

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Signal and Noise

In general, most data can be understood as observations of the form: signal + noise. This model envisions the observations as produced by a deterministic "signal" contaminated with random "noise". In data analysis, a model is fitted to the data, producing an "estimated signal", an the resulting residuals become the "estimated noise". The residuals, aka the estimated noise, are the basis for modeling uncertainty in the model. In most courses in data analysis, the focus is on white noise (independent, Normal errors, with zero mean and constant variance). The Time Series Analysis undertaken in this course will differ from Regression in that the noise has a complex structure that must be identified. A substantial part of the course will involve actually developing these more complex structures for the noise and then learning how to identify their presence in data.

Time Series Data

Following a long tradition, a set of observations collected over equally spaced points in time are called a time series. It should be noted that time series only require the special methods discussed in this course when the observations are serially correlated. So what is serial correlation? Serial correlation occurs when nearby observations are expected to be more similar than observations far apart (technically, serial correlation also occurs when nearby observations are more dissimilar than expected.)

White Noise (flavor 1)

A simple kind of generated series might be a collection of uncorrelated random variables, w_t , with mean 0 and finite variance σ_w^2. The time series generated from uncorrelated variables is used as a model for noise in engineering applications where it is called **white noise**; we shall sometimes denote this process as $w_t \sim wn(0, \sigma_w^2)$.

White Noise (flavor 2)

We will, at times, also require the noise to be independent and identically distributed (iid) random variables with mean 0 and variance σ_w^2. We will distinguish this by saying **white independent noise**, or by writing $w \sim iid(0, \sigma_w^2)$.

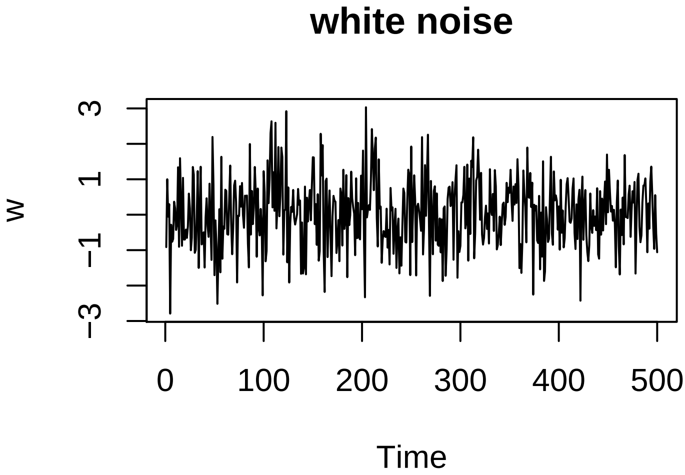
White Noise (flavor 3)

A particularly useful white noise series is **Gaussian white noise**, wherein the w_t are independent Normal random variables, with mean 0 and variance σ_w^2; or more succinctly, $w_t \sim iid\ N(0, \sigma_w^2)$.

R Code

```
set.seed(2016);
w=rnorm(500,0,1);
plot.ts(w, main="white noise");
```

Graph



Moving Averages and Filtering

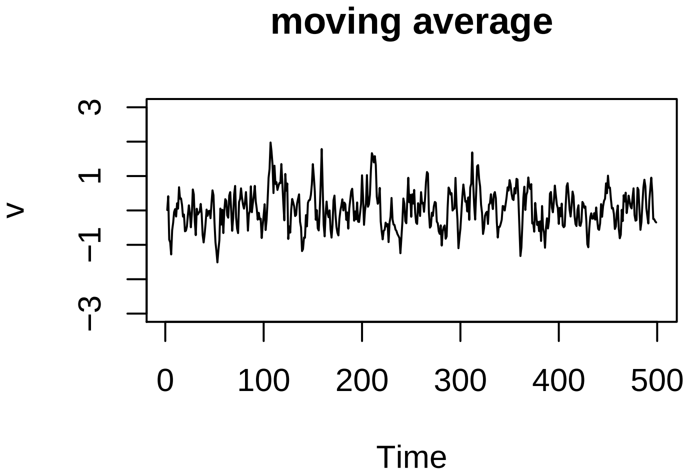
We might replace the white noise series w_t by a moving average that smooths the series. For example, consider replacing w_t in our previous example by an average of its current value and its immediate neighbors in the past and future. That is,

$$v_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1}).$$

(Note. A linear combination of values in a time series such as in $\frac{1}{3}(w_{t-1} + w_t + w_{t+1})$ is referred to as a filtered series.)

R Code

```
set.seed(2016);
w=rnorm(500,0,1);
v = filter(w, sides=2, filter=rep(1/3,3));
# v = moving average;
plot.ts(v, ylim=c(-3,3),main="moving average");
```


moving average
v
3
1
−1
−3
0
100
200
300
400
500
Time

Autoregressions

Suppose we consider the white noise series w_t as input and calculate the output using the first-order equation

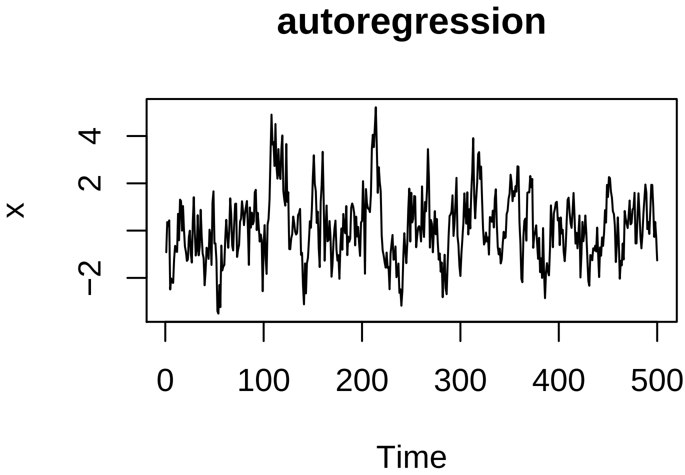
$$x_t = 0.7x_{t-1} + w_t$$

successively for $t = 1, 2, ..., 500$. The above equation represents a regression of the current value x_t of a time series as a function of the last value of the series, and, hence, the term autoregression is suggested for this model.

R Code

```r
set.seed(2016);
w = rnorm(500,0,1);
x = filter(w,filter=(0.7),method="recursive",init=0);
# method ="recursive" an autoregression is used;
# init = specifies initial values;
plot.ts(x,main="autoregression");
```

Graph



Random Walk with Drift

The random walk with drift model is given by

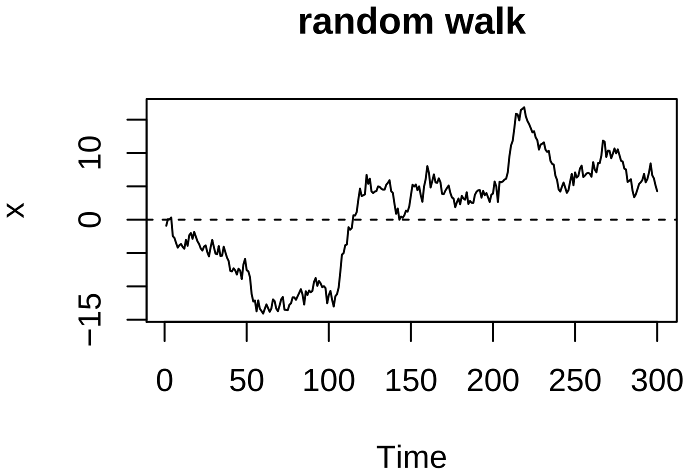
$$x_t = \delta + x_{t-1} + w_t$$

for $t = 1, 2, ...$, with initial condition $x_0 = 0$, and where w_t is white noise. The constant δ is called the drift. Note that we may rewrite x_t as a cumulative sum of white noise variates. That is, $x_t = \delta t + \sum_{j=1}^{t} w_j$ for $t = 1, 2, ...$

R Code

```
set.seed(2016);
w = rnorm(300,0,1);
x = cumsum(w);
# cumsum = cumulative sum;
plot.ts(x, main="random walk");
abline(h=0,lty=2);
# abline adds a horizontal line at zero;
# lty = 2, tells R to draw a dashed line;
```

R Code



Definition

The **mean function** is defined as

$$\mu_{xt} = E(x_t)$$

provided it exists, where E denotes the usual expected value operator. When no confusion exists about which time series we are referring to, we will drop a subscript and write μ_{xt} as μ_t.

Mean Function of a Moving Average Series

If w_t denotes a white noise series, then $\mu_{wt} = E(w_t) = 0$ for all t. Smoothing the series does not change the mean because we can write

$$\mu_{vt} = E(v_t) = \frac{1}{3}[E(w_{t-1}) + E(w_t) + E(w_{t+1})] = 0.$$

Mean function of a Random Walk with Drift

Consider the random walk with drift model

$$x_t = \delta t + \sum_{j=1}^{t} w_j, t = 1, 2, ...$$

Because $E(w_t) = 0$ for all t, and δ is a constant, we have

$$\mu_{xt} = E(x_t) = \delta t + \sum_{j=1}^{t} E(w_j) = \delta t$$

which is a straight line with slope δ.

Definition

The **autocovariance function** is defined as the second moment product

$$\gamma_x(s,t) = cov(x_s, x_t) = E[(x_s - \mu_s)(x_t - \mu_t)],$$

for all s and t. When no possible confusion exists about which time series we are referring to, we will drop the subscript and write $\gamma_x(s,t)$ as $\gamma(s,t)$. It is clear that, for $s = t$, the autocovariance reduces to the variance, because

$$\gamma_x(t,t) = E[(x_t - \mu_t)^2] = var(x_t).$$

Autocovariance of White Noise

The white noise series w_t has $E(w_t) = 0$ and

$$\gamma_w(s,t) = cov(w_s, w_t) = \begin{cases} \sigma_w^2 & s = t, \\ 0 & s \neq t. \end{cases}$$

Proposition

If the random variables
$U = \sum_{j=1}^{m} a_j X_j$ and $V = \sum_{k=1}^{r} b_k Y_k$
are linear filters of random variables $\{Xj\}$ and $\{Y_k\}$, respectively, then

$$cov(U, V) = \sum_{j=1}^{m} \sum_{k=1}^{r} a_j b_k cov(X_j, Y_k).$$

Furthermore, $var(U) = cov(U, U)$.

Autocovariance of a Moving Average

Consider applying a three-point moving average to the white noise series w_t of the previous example. In this case,

$$\gamma_v(s,t) = cov(v_s, v_t) = cov\left[\frac{1}{3}(w_{s-1} + w_s + w_{s+1}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right].$$

Autocovariance of a Moving Average

When $s = t$ we have

$$
\begin{aligned}
\gamma_v(t,t) &= cov\left[\frac{1}{3}(w_{t-1} + w_t + w_{t+1}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right] \\
&= \frac{1}{9}\left[cov(w_{t-1}, w_{t-1}) + cov(w_t, w_t) + cov(w_{t+1}, w_{t+1})\right] \\
&= \frac{3}{9}\sigma_w^2.
\end{aligned}
$$

Autocovariance of a Moving Average

When $s = t + 1$ we have

$$\begin{aligned}\gamma_v(t+1,t) &= \frac{1}{9}cov\left[(w_t + w_{t+1} + w_{t+2}),(w_{t-1} + w_t + w_{t+1})\right] \\ &= \frac{1}{9}\left[cov(w_t, w_t) + cov(w_{t+1}, w_{t+1})\right] \\ &= \frac{2}{9}\sigma_w^2.\end{aligned}$$

Autocovariance of a Moving Average

Similar computations give $\gamma_v(t-1,t)$, $\gamma_v(t+2,t)$, $\gamma_v(t-2,t)$, and 0 when $|t-s|>2$. We summarize the values for all s and t as

$$\gamma_v(s,t) = \begin{cases} \frac{3}{9}\sigma_w^2 & s = t, \\ \frac{2}{9}\sigma_w^2 & |s-t| = 1, \\ \frac{1}{9}\sigma_w^2 & |s-t| = 2, \\ 0 & |s-t| > 2. \end{cases}$$

Autocovariance of a Random Walk

For the random walk model, $x_t = \sum_{j=1} w_j$, we have

$$\gamma_x(s,t) = cov(x_s, x_t) = cov\left(\sum_{j=1}^{s} w_j, \sum_{k=1}^{t} w_k\right) = min\{s,t\}\sigma_w^2,$$

because the w_t are uncorrelated random variables.

Definition

The **autocorrelation function (ACF)** is defined as

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}.$$

Definition

A **weakly stationary** time series is a finite variance process where
i) the mean value function, μ_t, is constant and does not depend on time t, and
ii) the autocovariance function, $\gamma(s, t)$, depends on s and t only through their difference $|s - t|$.
Henceforth, we will use the term **stationary** to mean weakly stationary.

Random Walk (again...)

A random walk **is not** stationary because its covariance function $\gamma(s,t) = min\{s,t\}\sigma_w^2$, depends on time. Also, the random walk with drift violates both conditions of the definition of a weakly stationary time series because the mean function, $\mu_x t = \delta t$, is also a function of time t.

Definition

The **autocovariance function of a stationary time series** will be written as

$$\gamma(h) = cov(x_{t+h}, x_t) = E[(x_{t+h} - \mu)(x_t - \mu)].$$

Definition

The **autocorrelation function (ACF) of a stationary time series** will be written as

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}.$$

White Noise (again...)

The mean and autocovariance functions of the white noise series discussed above are easily evaluated as $\mu_{wt} = 0$ and

$$\gamma_w(h) = cov(w_{t+h}, w_t) = \begin{cases} \sigma_w^2 & h = 0, \\ 0 & h \neq 0. \end{cases}$$

Thus, white noise is weakly stationary or stationary.

A Moving Average (again...)

The three-point moving average process presented above is stationary because the mean and autocovariance functions $\mu_{vt} = 0$, and

$$\gamma_v(h) = \begin{cases} \frac{3}{9}\sigma_w^2 & h = 0, \\ \frac{2}{9}\sigma_w^2 & h = \pm 1, \\ \frac{1}{9}\sigma_w^2 & h = \pm 2, \\ 0 & |h| > 2. \end{cases}$$

are independent of time t, satisfying the conditions of a weakly stationary time series.

Moving Average (ACF)

$$\rho_v(h) = \begin{cases} 1 & h = 0, \\ \frac{2}{3} & h = \pm 1, \\ \frac{1}{3} & h = \pm 2, \\ 0 & |h| > 2. \end{cases}$$

Trend Stationarity

If $x_t = \alpha + \beta t + w_t$, then the mean function is $\mu_{x,t} = E(x_t) = \alpha + \beta t$, which is not independent of time. Therefore, the process is not stationary. The autocovariance function, however, is independent of time, because $\gamma_w(h) = cov(x_{t+h}, x_t) = E[(x_{t+h} - \mu_{x,t+h})(x_t - \mu_{x,t})] = E(w_{t+h}w_t) = \gamma_w(h)$.
Thus, the model may be considered as having stationary behavior around a linear trend; this behavior is sometimes called **trend stationarity**.