ARMA Models

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Definition

$\{x_t\}$ is an ARMA(p,q) process if $\{x_t\}$ is stationary and if for every t,

$$x_t - \phi_1 x_{t-1} - ... - \phi_p x_{t-p} = w_t + \theta_1 w_{t-1} + ... + \theta_q w_{t-q}$$

where $\{w_t\}$ is white noise with mean 0 and variance σ_w^2 and the polynomials $1 - \phi_1 z - ... - \phi_p z^p$ and $1 + \theta_1 z + ... + \theta_q z^q$ have no common factors.

Example. ACF of the ARMA(1,1) Process

Model:

$$x_t = \phi_1 x_{t-1} - \theta_1 w_{t-1} + w_t$$

For stationarity, we assume $|\phi_1| < 1$, and for invertibility, we require that $|\theta_1| < 1$ .We also assume that $E(x_t) = 0$ and $E(w_t) = 0$.

Example. ACF of the ARMA(1,1) Process

We showed in class that

$$\begin{aligned}
\gamma(0) &= \frac{(1+\theta_1^2-2\phi_1\theta_1)\sigma_w^2}{1-\phi_1^2} \\
\gamma(1) &= \frac{(\phi_1-\theta_1)(1-\phi_1\theta_1)\sigma_w^2}{1-\phi_1^2} \\
\gamma(k) &= \phi_1\gamma(k-1) \text{ for } k \geq 2.
\end{aligned}$$

Example. ACF of the ARMA(1,1) Process

We showed in class that

$$
\begin{aligned}
\rho(0) &= 1 \\
\rho(1) &= \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1 + \theta_1^2 - 2\phi_1\theta_1} \\
\rho(k) &= \phi_1 \rho(k-1) \text{ for } k \geq 2.
\end{aligned}
$$

Note that the autocorrelation function of an ARMA(1,1) model combines characteristics of both AR(1) and MA(1) processes.

Causality

An ARMA(p, q) process $\{x_t\}$ is causal, or a casual function of $\{w_t\}$, if there exist constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} \quad \text{for all } t.$$

Invertibility

An ARMA(p, q) process $\{x_t\}$ is invertible if there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} \quad \text{for all } t.$$

Definition

The AR and MA polynomials are defined as

$$\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p, \quad \phi_p \neq 0$$

and

$$\theta(z) = 1 + \theta_1 z - ... + \theta_q z^q, \quad \theta_q \neq 0,$$

respectively, where z is a complex number.

Example

Consider the process

$$x_t = 0.4x_{t-1} + 0.45x_{t-2} + w_t + w_{t-1} + 0.25w_{t-2}$$

or, in operator form,

$$(1 - 0.4B - 0.45B^2)x_t = (1 + B + 0.25B^2)w_t$$

(it seems to be an ARMA(2, 2) process).

Example

Note that $\phi(z) = (1 + 0.5z)(1 - 0.9z)$ and $\theta(z) = (1 + 0.5z)(1 + 0.5z)$. So, the associated polynomials have a common factor $(1 + 0.5z)$ that can be canceled. After cancellation, the polynomials become

$$\phi(z) = (1 - 0.9z)$$

and

$$\theta(z) = (1 + 0.5z),$$

so the model is an ARMA(1, 1) process.

Example

The model is **causal** because the root of $\phi(z)$ is $s = \frac{10}{9}$, which is outside the unit circle (i. e. $|s| > 1$).

The model is **invertible** because the root of $\theta(z)$ is $s_* = -2$, which is also outside the unit circle.

Another example

Let $\{x_t\}$ be the AR(2) process

$$x_t = 0.7x_{t-1} - 0.1x_{t-2} + w_t,$$

or, in operator form,

$$(1 - 0.7B + 0.1B^2)x_t = w_t$$

Another example (cont.)

The corresponding polynomial is

$$\phi(z) = 1 - 0.7z + 0.1z^2 = c + bz + az^2.$$

Its roots can be found using the following formula

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0.7 \pm 0.3}{0.2}.$$

So, $s_1 = 5$ and $s_2 = 2$. Since these "zeros" lie outside the unit circle, we conclude that $\{x_t\}$ is a **causal** AR(2) process.

Quick Review of Complex Numbers

The number i.
Complex numbers can be loosely considered linear combinations of real and imaginary numbers. The core of imaginary numbers is the quantity $i = \sqrt{-1}$. Any number of the form $a + bi$, where a and b are real, is a complex number.
Note that:

$$i^2 = (\sqrt{-1})^2 = -1$$

$$i^3 = (\sqrt{-1})^3 = -i$$

$$i^4 = (\sqrt{-1})^4 = 1$$

, etc.

Sums, differences, products, and quotients of complex numbers are also complex numbers.

Examples:

$$(3 - 2i) + (3 - 7i) = 6 - 9i$$

$$(3 - i) + (7 + 4i) = -4 - 5i$$

$$(1 - 2i)(2 - i) = 2 - i - 4i + 2i^2 = 2 - 5i - 2 = -5i$$

Sums, differences, products, and quotients of complex numbers are also complex numbers.

$$\frac{3-2i}{1-2i} = \frac{(3-2i)(1+2i)}{(1-2i)(1+2i)} = \frac{3+4i+4}{1^2+2^2} = \frac{7+4i}{5}$$

$$\frac{3-2i}{1-2i} = \frac{7}{5} + \frac{4i}{5}$$

Notation. Let $z = a + bi$, then $\bar{z} = a - bi$.

The magnitude of a complex number

Complex conjugates can be used to represent magnitude. Let $z = a + bi$ and use $|z|$ to represent magnitude. Then $|z|^2 = z\bar{z} = a^2 + b^2$ and $|z| = \sqrt{a^2 + b^2}$.

Example

Consider the process defined by the equations

$$x_t - 0.75x_{t-1} + 0.5625x_{t-2} = w_t + 1.25w_{t-1}$$

where w_t represents white noise with mean 0 and variance σ_w^2.

Example (cont.)

The AR polynomial:

$$\phi(z) = 1 - 0.75z + 0.5625z^2.$$

Its roots are given by:

$$z^* = \frac{0.75 \pm \sqrt{(-0.75)^2 - 4(0.5625)(1)}}{2(0.5625)}$$

$$z^* = \frac{0.75 \pm 1.2990}{1.125} \approx 0.6666 \pm 1.1547i$$

$$|z^*| = \sqrt{0.6666^2 + 1.1547^2} \approx 1.3332 > 1,$$

which lies outside the unit circle. The process is therefore causal.

Example (cont.)

On the other hand, the MA polynomial is

$$\theta(z) = 1 + 1.25z.$$

Clearly, $z^* = -0.8$ and $|z^*| < 1$. Hence $\{x_t\}$ is **not** invertible.

Simulated Examples of ARMA Models

Example 1. ARMA(2,2):

$$x_t = 0.6x_{t-1} - 0.25x_{t-2} + w_t + 1.1w_{t-1} - 0.28w_{t-2}.$$

Example 2. ARMA(2,2):

$$x_t = 1.1x_{t-1} - 0.28x_{t-2} + w_t + 0.6w_{t-1} - 0.25w_{t-2}.$$

Example 3. ARMA(3,0):

$$x_t = 0.6x_{t-1} - 0.19x_{t-2} + 0.084x_{t-3} + w_t.$$

Example 4. ARMA(0,4):

$$x_t = w_t + 2w_{t-1} - 1.59w_{t-2} + 0.65w_{t-3} - 0.125w_{t-4}.$$

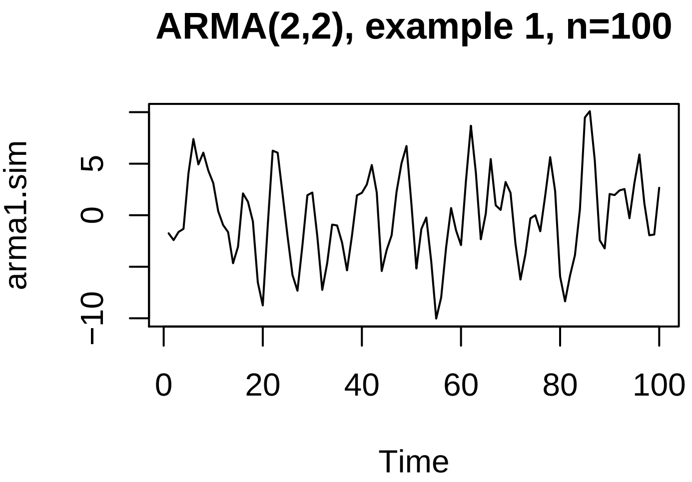
R Code, example 1

```
set.seed(9999);

# simulating ARMA(2,2);
arma1.sim<-arima.sim(list(ar=c(0.6,-0.25),
ma = c(1.1,-0.28)), n = 100, sd=2);

plot.ts(arma1.sim, ylim=c(-10,10),main="ARMA(2,2),
example 1, n=100");
```

Time Series Plot, example 1



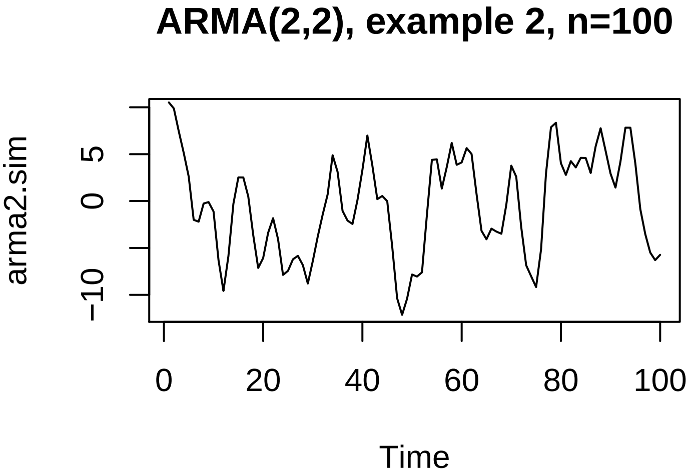
R Code, example 2

```
set.seed(9999);

# simulating ARMA(2,2);
arma2.sim<-arima.sim(list(ar=c(1.1,-0.28),
ma = c(0.6,-0.25)), n = 100, sd=2);

plot.ts(arma2.sim, ylim=c(-12,10),main="ARMA(2,2),
example 1, n=100");
```

Time Series Plot, example 2



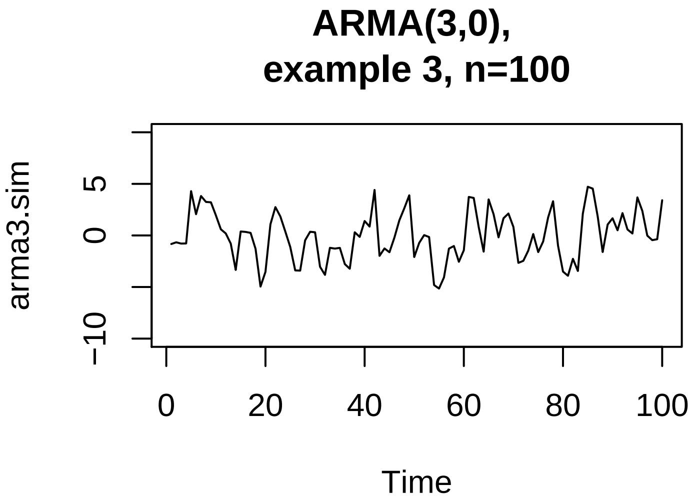
R Code, example 3

```
set.seed(9999);

# simulating AR(3);
arma3.sim<-arima.sim(list(ar=c(0.6,-0.19, 0.084) ),
n = 100, sd=2);

plot.ts(arma3.sim, ylim=c(-10,10),main="ARMA(3,0),
example 3, n=100");
```

Time Series Plot, example 3



R Code, example 4

```
set.seed(9999);

# simulating ARMA(0,4);
arma4.sim<-arima.sim(list(ma = c(2,-1.59,0.65,-0.125) ),
n = 100, sd=2);

plot.ts(arma4.sim, ylim=c(-15,15),main="ARMA(0,4),
example 4, n=100");
```

Time Series Plot, example 4