TUTORIAL 7
STA437 WINTER 2015

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Contents

1. Tests on Covariance Matrices

1.1. **Testing** $H_0 : \mathbf{\Sigma} = \mathbf{\Sigma_0}$**.** We begin with the basic hypothesis $H_0 : \mathbf{\Sigma} = \mathbf{\Sigma_0}$ versus $H_1 : \mathbf{\Sigma} \neq \mathbf{\Sigma_0}$. The hypothesized covariance matrix $\mathbf{\Sigma_0}$ is a target value for $\mathbf{\Sigma_0}$ or a nominal value from previous experience. To test H_0, we obtain a random sample of n observation vectors $\mathbf{y_1}, \mathbf{y_2}, ..., \mathbf{y_n}$ from $N_p(\mu, \mathbf{\Sigma})$ and calculate $\mathbf{S}$. To see if $\mathbf{S}$ is significantly different from $\mathbf{\Sigma_0}$, we use the following test statistic, which is a modification of the likelihood ratio

$$u = \nu[ln|\mathbf{\Sigma_0}| - ln|\mathbf{S}| + tr(\mathbf{S}\mathbf{\Sigma_0}^{-1}) - p]$$

(in one of our lectures, we showed that $u = -2ln(\lambda)$, where λ represents a likelihood ratio test)

where ν represents the degrees of freedom of $\mathbf{S}$. For a single sample, $\nu = n - 1$; for a pooled covariance matrix, $\nu = \sum_{i=1}^{k} n_i - k = N - k$. Note that if $\mathbf{S} = \mathbf{\Sigma_0}$, then $u = 0$; otherwise u increases with the "distance" between $\mathbf{S}$ and $\mathbf{\Sigma_0}$. When ν is large, the statistic u is approximately distributed as $\chi^2[\frac{1}{2}p(p+1)]$ if H_0 is true. For moderate size ν,

$$u' = \left[1 - \frac{1}{6\nu - 1}\left(2p + 1 - \frac{2}{p+1}\right)\right] u$$

is a better approximation to the $\chi^2[\frac{1}{2}p(p+1)]$ distribution. We reject H_0 if u or u' is greater than $\chi^2[\alpha, \frac{1}{2}p(p+1)]$. Note that the degrees of freedom for the χ^2-statistic, $\frac{1}{2}p(p+1)$, is the number of distinct parameters in $\mathbf{\Sigma}$.

We can express u in terms of the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$ of $\mathbf{S}\mathbf{\Sigma_0}^{-1}$, then u can be written as

$$u = \nu \left[\sum_{i=1}^{p} (\lambda_i - ln(\lambda_i) - p \right].$$

Exercise 7.14. In Example 5.2.2, height and weight were given for a sample of 20 college-age males, we assumed that for the height and weight data of this example, the population covariance matrix is

$$\mathbf{\Sigma} = \begin{pmatrix} 20 & 100 \\ 100 & 1000 \end{pmatrix},$$

the sample covariance matrix is

$$\mathbf{S} = \begin{pmatrix} 14.58 & 128.87 \\ 128.870 & 1441.27 \end{pmatrix}.$$

Test this as a hypothesis using $u^{'}$ (Use $\alpha = 0.01$).

Solution

To find u and $u^{'}$ we need: ν, p, $|\mathbf{\Sigma_0}|$, $|\mathbf{S}|$, $tr(\mathbf{S}\mathbf{\Sigma_0}^{-1})$. In this case $\nu = n - 1 = 20 - 1 = 19$, $p = 2$, $|\mathbf{\Sigma_0}| = 10000$, $|\mathbf{S}| = 4406.24$, and $tr(\mathbf{S}\mathbf{\Sigma_0}^{-1}) = 1.76314$. Which yield

$$u = 11.07$$

$$u^{'} = 0.96166u = 10.64$$

Our critical value is given by

$$\chi^2[\alpha, \frac{1}{2}p(p+1)] = \chi^2[0.01, 3] = 11.345$$

We **can't** reject the hypothesized covariance matrix.

Example. Let us suppose that the reaction times in hundredths of a second after three preparatory intervals can be described by a trivariate Normal random variable. Reaction times have been measured under the three conditions on a random sample of $N = 20$ normal subjects. From those data we wish to test the hypothesis H_0 :

$$\mathbf{\Sigma} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 6 & 5 \\ 2 & 5 & 10 \end{pmatrix}$$

that has been suggested by the effects of the lengthening preparatory intervals on the variances and correlations, as well as by previous experimental results. The sample covariance is

$$\mathbf{S} = \begin{pmatrix} 3.42 & 2.60 & 1.89 \\ 2.60 & 8 & 6.51 \\ 1.89 & 6.51 & 9.62 \end{pmatrix}$$

(Use $\alpha = 0.05$).

Solution

To find u and $u^{'}$ we need: ν, p, $|\mathbf{\Sigma_0}|$, $|\mathbf{S}|$, $tr(\mathbf{S}\mathbf{\Sigma_0}^{-1})$. In this case $\nu = n - 1 = 20 - 1 = 19$, $p = 3$, $|\mathbf{\Sigma_0}| = 86$, $|\mathbf{S}| = 88.6355$, and $tr(\mathbf{S}\mathbf{\Sigma_0}^{-1}) = 3.2216$. Which yield

$$u = 3.64$$

$$u^{'} = 3.43$$

Our critical value is given by

$$\chi^2[\alpha, \frac{1}{2}p(p+1)] = \chi^2[0.05, 3] = 12.592$$

Since $u^{'}$ does not exceed our critical value, we conclude that the hypothesized covariance matrix is tenable.

R code

```
Sigma.0<-matrix(c(4,3,2,3,6,5,2,5,10),nrow=3,ncol=3)

Sigma.0

Sigma.0.inv<-solve(Sigma.0)

Sigma.0.inv

S<-matrix(c(3.42,2.60,1.89,2.60,8,6.51,1.89,6.51,9.62),nrow=3,ncol=3)

S

S%*%Sigma.0.inv


trace<-sum(diag(S%*%Sigma.0.inv))

trace
```

```
##########################
## test statistic  u
##########################

# n = number of individuals

n<-20

# p = dimension of your data

p<-3

# nu

nu<-n-1

u<-nu*( log(det(Sigma.0)) - log(det(S))  + trace -p )

u

##########################
###  test statistic u'
##########################

# k =constant

k<-1-(2*p+1 - (2/(p+1)))*(1/(6*nu-1))

u.star<-k*u

u.star
```

1.2. **Testing Sphericity.** The hypothesis that the variables $y_1, y_2, ..., y_p$ in $\mathbf{y}$ are independent and have the same variance can be expressed as $H_0 : \mathbf{\Sigma} = \sigma^2\mathbf{I}$ versus $H_1 : \mathbf{\Sigma} \neq \sigma^2\mathbf{I}$, where σ^2 is the unknown common variance. This hypothesis is of interest in repeated measures. Under H_0, the ellipsoid $(\mathbf{y} - \mu)'\mathbf{\Sigma^{-1}}(\mathbf{y} - \mu) = c^2$ reduces to $(\mathbf{y} - \mu)'(\mathbf{y} - \mu) = \sigma^2c^2$, the equation of a sphere; hence the term **sphericity** is applied to the covariance structure $\mathbf{\Sigma} = \sigma^2\mathbf{I}$. Another sphericity hypothesis of interest in repeated measures is $H_0 : \mathbf{C\Sigma C}' = \sigma^2\mathbf{I}$, where $\mathbf{C}$ is any full-rank $(p - 1) \times p$ matrix of orthonormal contrasts. For a random sample $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n$ from $N_p(\mu, \mathbf{\Sigma})$, the likelihood ratio for testing $H_0 : \mathbf{\Sigma} = \sigma^2\mathbf{I}$ is

$$LR = \left[\frac{|\mathbf{S}|}{(tr\mathbf{S}/p)^p}\right]^{n/2}$$

It has been shown that for a general likelihood ratio statistic LR,

$$-2ln(LR) \text{ is approximately } \chi^2_\nu$$

for large n, where ν is the total number of parameters minus the number of estimated under the restrictions imposed by H_0. In this case, we obtain

$$-2ln(LR) = -nln\left[\frac{|\mathbf{S}|}{(tr\mathbf{S}/p)^p}\right] = -n\ ln\ u$$

where $u = \frac{p^p|\mathbf{S}|}{(tr\mathbf{S})^p}$. We can express u in terms of the eigenvalues of $\mathbf{S}$, doing so yields

$$u = \frac{p^p \Pi_{i=1}^p \lambda_i}{(\sum_{i=1}^p \lambda_i)^p}.$$

An improvement over $-n\ ln\ u$ is given by

$$u^{'} = -\left(\nu - \frac{2p^2+p+2}{6p}\right) ln\ u,$$

where ν is the degrees of freedom for $\mathbf{S}$. The statistic $u^{'}$ has an approximate χ^2-distribution with $\frac{1}{2}p(p+1)-1$ degrees of freedom. We reject H_0 if $u^{'} \geq \chi^2[\alpha, \frac{1}{2}p(p+1)-1]$. To test $H_0 : \mathbf{C\Sigma C}^{'} = \sigma^2\mathbf{I}$, use $\mathbf{C\Sigma C}^{'}$ in place of $\mathbf{S}$ and use $p-1$ in place of p, including in the degrees of freedom for χ^2

$$u = \frac{(p-1)^{(p-1)}|\mathbf{S}|}{(tr\mathbf{S})^{(p-1)}}$$

$$u^{'} = -\left(\nu - \frac{2p^2-3p+3}{6(p-1)}\right) ln\ u,$$

.

Example. We use the probe word data in word.txt to illustrate tests of sphericity. The five variables appear to be commensurate, an the hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$ may be of interest. We would expect the variables to be correlated, and H_0 would ordinarily be tested using a multivariate approach. However, if $\mathbf{\Sigma} = \sigma^2\mathbf{I}$ or $\mathbf{C\Sigma C}^{'} = \sigma^2\mathbf{I}$, then the hypotheses $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$ can be tested with a univariate ANOVA F-test. We first test $H_0 : \mathbf{\Sigma} = \sigma^2\mathbf{I}$.

$$u = \frac{p^p|\mathbf{S}|}{(tr\mathbf{S})^p} = \frac{5^5(27,236,586)}{(292.891)^5} = 0.0395$$

(with $n = 11$ and $p = 5$)

$$u^{'} = -\left(\nu - \frac{2p^2+p+2}{6p}\right) ln\ u = 26.177$$

The approximate χ^2-test has $\frac{1}{2}p(p+1)-1 = 14$ degrees of freedom. We therefore compare $u^{'} = 26.177$ with $\chi^2_{0.05,14} = 23.685$ and **reject** $H_0 : \mathbf{\Sigma} = \sigma^2\mathbf{I}$. To test $H_0 : \mathbf{C\Sigma C}^{'} = \sigma^2\mathbf{I}$, we use the following matrix of orthonormalized contrasts:

$$\mathbf{C} = \begin{pmatrix} 4/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{20} \\ 0 & 3/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} & -1/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

We obtain

$$u = \frac{(p-1)^(p-1)|\mathbf{S}|}{(tr\mathbf{S})^(p-1)} = \frac{4^4(144039.8)}{(93.6)^4} = 0.480,$$

$$u^{'} = 6.183.$$

For degrees of freedom, we now have $\frac{1}{2}(4)(5)-1 = 9$, and the critical value is $\chi^2_{0.05,9} = 16.919$. Hence, we **do not** reject $\mathbf{C\Sigma C}^{'} = \sigma^2\mathbf{I}$, and a univariate F-test of $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$ may be justified.

R code

```
word<-read.table(file="word.txt")

word

S<-cov(word)

S

n<-dim(word)[1]

n

p<-dim(word)[2]

p

## test statistic
```

```
denom<-( sum(diag(S/p)) )^p

LR<-(det(S)/denom)^(n/2)

u<-(LR)^(2/n)

u

##

nu<-n-1

## test statistic u'

# k = constant

k<-(-1)*(nu-1*(2*p^2+p+2)/(6*p))

u.star<-k*log(u)

u.star

## critical value

alpha<-0.05

## DF= degrees of freedom

DF<-(0.5)*p*(p+1)-1

crit.val<-qchisq(1-alpha,DF)

crit.val

### CSC'

## C = matrix of contrasts

v<-rev(contr.helmert(5))
```

```
C<-matrix(v,nrow=4,ncol=5,byrow=TRUE)

## normalizing C

C<-C/diag(sqrt(C%*%t(C)))

## test statistic u

u<-(p-1)^(p-1)*det(C%*%S%*%t(C))/( sum(diag(C%*%S%*%t(C)))  )^(p-1)

u

## test statistic u'

## k = constant

k<- (-1)*(nu - ( 2*p^2-3*p+3 )/( 6*(p-1) ) )

u.star<-k*log(u)

u.star

## critical value

alpha<-0.05

## DF= degrees of freedom

DF<-(0.5)*(p-1)*(p)-1

crit.val<-qchisq(1-alpha,DF)

crit.val
```

Exercise 7.17. Test $H_0 : H_0 : \mathbf{\Sigma} = \sigma^2\mathbf{I}$ and $H_0 : \mathbf{C\Sigma C}' = \sigma^2\mathbf{I}$ for the cork data.