STA 260: Statistics and Probability II

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"If you can't explain it simply, you don't understand it well enough"

Albert Einstein.

Example

Let (Y_1, Y_2) denote a random sample of size $n = 2$ from the uniform distribution on the interval $(0, 1)$. Find the probability density function for $U = Y_1 + Y_2$.

Solution

The density function for each Y_i is

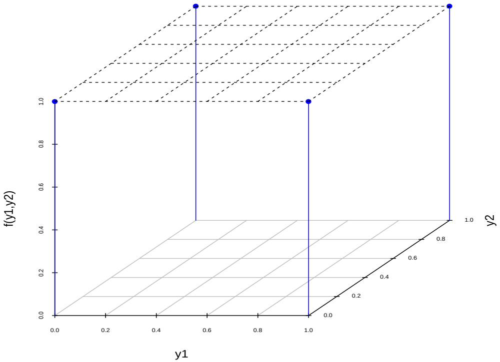
$$f(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & elsewhere \end{cases}$$

Therefore, because we have a random sample, Y_1 and Y_2 are independent, and

$$f(y_1, y_2) = f(y_1)f(y_2) \begin{cases} 1 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & elsewhere \end{cases}$$

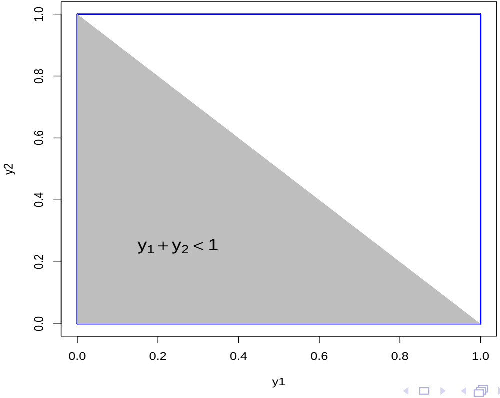
We wish to find $F_U(u) = P(U \leq u)$.

Joint pdf



Solution

The region $y_1 + y_2 \leq u$ for $0 \leq u \leq 1$.



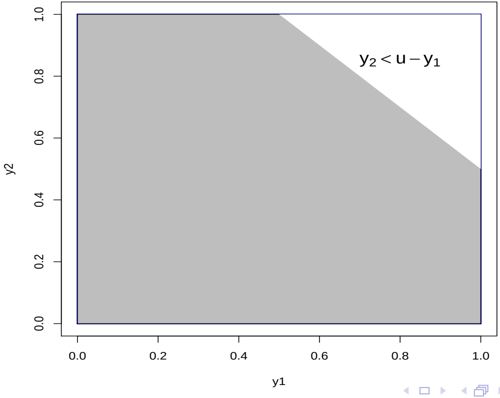
Solution

The solution, $F_U(u), 0 \leq u \leq 1$, could be acquired directly by using elementary geometry.

$F_U(u) = (\text{area of triangle})(\text{height}) = \frac{u^2}{2}(1) = \frac{u^2}{2}$.

Solution

The region $y_1 + y_2 \leq u$ for $1 < u \leq 2$.



Solution

The solution, $F_U(u), 1 < u \leq 2$, could be acquired directly by using elementary geometry, or using Calculus.

$$\begin{aligned} F_U(u) &= 1 - (\text{area of triangle})(\text{height}) \\ &= 1 - \left[\frac{(2-u)(2-u)}{2}\right](1) \\ &= 1 - \left[2 - 2u + \frac{u^2}{2}\right] \\ &= -1 + 2u - \frac{u^2}{2} \end{aligned}$$

Solution

It should be clear at this point that
If $u < 0$, $F_U(u) = 0$.
If $u > 2$ $F_U(u) = 1$.

Solution

To summarize,

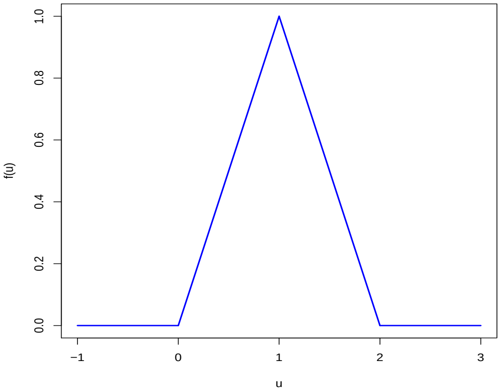
$$F_U(u) = \begin{cases} 0 & u \leq 0 \\ u^2/2 & 0 < u \leq 1 \\ (-u^2/2) + 2u - 1 & 1 < u \leq 2 \\ 1 & u > 2 \end{cases}$$

Solution

The density function $f_U(u)$ can be obtained by differentiating $F_U(u)$. Thus,

$$f_U(u) = \frac{d\ F_U(u)}{du} = \begin{cases} 0 & u \leq 0 \\ u & 0 \leq u \leq 1 \\ 2 - u & 1 < u \leq 2 \\ 0 & u > 2 \end{cases}$$

Graph of pdf



Example

Consider the case $U = h(Y) = Y^2$, where Y is a continuous random variable with distribution function $F_Y(y)$ and density function $f_Y(y)$. Find the probability density function for U.

Solution

If $u \leq 0$,
$F_U(u) = P(U \leq u) = P(Y^2 \leq u) = 0$.
If $u > 0$,
$F_U(u) = P(U \leq u) = P(Y^2 \leq u) = P(-\sqrt{u} \leq Y \leq \sqrt{u})$
$= \int_{-\sqrt{u}}^{\sqrt{u}} f(y)dy = F_Y(\sqrt{u}) - F_Y(-\sqrt{u})$.

Solution

On differentiating with respect to u, we see that

$$f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}}[f_Y(\sqrt{u}) + f_Y(-\sqrt{u})] & u > 0 \\ 0 & otherwise \end{cases}$$

Exercise 6.7

Suppose that Z has a standard Normal distribution.
a. Find the density function of $U = Z^2$.
b. Does U have a gamma distribution? What are the values of α and β?
c. What is another name for the distribution of U?

Solution

Let $F_Z(z)$ and $f_Z(z)$ denote the standard Normal distribution and density functions respectively.

a. $F_U(u) = P(U \leq u) = P(Z^2 \leq u) = P(-\sqrt{u} \leq Z \leq \sqrt{u})$
$= F_Z(\sqrt{u}) - F_Z(-\sqrt{u})$.

The density function for U is then

$f_U(u) = F_U^{'}(u) = \frac{1}{2\sqrt{u}} f_Z(\sqrt{u}) + \frac{1}{2\sqrt{u}} f_Z(-\sqrt{u}),\ u \geq 0.$

Solution

Recalling that $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, we find

$$f_U(u) = \frac{1}{2\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} + \frac{1}{2\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}}$$

$$f_U(u) = \frac{1}{\sqrt{\pi}\sqrt{2}} u^{-1/2} e^{-u/2}, \; u > 0.$$

Solution

b. U has a gamma distribution with $\alpha = 1/2$ and $\beta = 2$ (recall that $\Gamma(1/2) = \sqrt{\pi}$).
c. This is the chi-square distribution with one degree of freedom.

Weibull density function

The Weibull density function is given by

$$f(y) = \begin{cases} \frac{1}{\alpha} m y^{m-1} e^{-y^m/\alpha} & y > 0, \\ 0, & elsewhere, \end{cases}$$

where α and m are positive constants. This density function is often used as a model for the lengths of life of physical systems.

Exercise 6.27

Let Y have an exponential distribution with mean β. Prove that $W = \sqrt{Y}$ has a Weibull density with $\alpha = \beta$ and $m = 2$.

Solution

Let $W = \sqrt{Y}$. The random variable Y is exponential so $f_Y(y) = \frac{1}{\beta} e^{-y/\beta}$.

Step 1. Then, $Y = W^2$.

Step 2. $\frac{dy}{dw} = 2w$.

Step 3. Then,

$$f_W(w) = f_Y(w^2)|2w| = \left(\frac{1}{\beta} e^{-w^2/\beta}\right)(2w) = \frac{2}{\beta} w e^{-w^2/\beta},\ w \geq 0,$$

which is Weibull with $m = 2$.

Exercise 6.28

Let Y have a uniform $(0, 1)$ distribution. Show that $W = -2ln(Y)$ has an exponential distribution with mean 2.

Solution

Step 1. Then, $Y = e^{-w/2}$.
Step 2. $\frac{dy}{dw} = \frac{-1}{2} e^{-w/2}$.
Step 3. Then, $f_W(w) = f_Y(e^{-w/2}) |\frac{-1}{2} e^{-w/2}| = \frac{1}{2} e^{-w/2}, w > 0$.

Exercise 6.29 a.

The speed of a molecule in a uniform gas at equilibrium is a random variable V whose density function is given by $f(v) = av^2 e^{-bv^2}$, $v > 0$, where $b = m/2kT$ and k, T, and m denote Boltzmann's constant, the absolute temperature, and the mass of the molecule, respectively.
Derive the distribution of $W = mV^2/2$, the kinetic energy of the molecule.

Solution

Step 1. With $W = \frac{mV^2}{2}$, $V = \sqrt{\frac{2W}{m}} = \left(\frac{2W}{m}\right)^{1/2}$.

Step 2. $|\frac{dv}{dw}| = |(\frac{1}{2})(\frac{2W}{m})^{-1/2}(\frac{2}{m})| = |\frac{1}{\sqrt{2mw}}|$.

Step 3. Then, $f_W(w) = f_V(\sqrt{\frac{2W}{m}})|\frac{1}{\sqrt{2mw}}| =$
$a(2w/m)e^{-b(2w/m)}\frac{1}{\sqrt{2mw}} = \frac{a\sqrt{2}}{m^{3/2}}w^{1/2}e^{-w/kT}$, $w > 0$.

Solution

The above expression looks like a Gamma density with $\alpha = 3/2$ and $\beta = kT$. Thus, the constant a must be chosen so that

$$\frac{a\sqrt{2}}{m^{3/2}} = \frac{1}{\Gamma(3/2)(KT)^{3/2}}.$$

So,

$$f_W(w) = \frac{1}{\Gamma(3/2)(KT)^{3/2}} w^{1/2} e^{-w/kT}.$$

Example

Let Z be a Normally distributed random variable with mean 0 and variance 1. Use the method of moment-generating functions to find the probability distribution of Z^2.

Solution

$M_{Z^2}(t) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$
$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2\left(\frac{1-2t}{2}\right)} dz$

This integral can be evaluated using an "old trick" (we note that it looks like a Normally distributed random variable).

Solution

We realize that $e^{-z^2(\frac{(1-2t)}{2})}$ is proportional to a Normal with $\mu = 0$ and $\sigma^2 = 1/(1-2t)$, then

$M_{Z^2}(t) = \frac{\sqrt{2\pi}\sqrt{1/(1-2t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1/(1-2t)}} e^{-z^2(\frac{1-2t}{2})} dz$

$M_{Z^2}(t) = \sqrt{\frac{1}{1-2t}} = (1-2t)^{-1/2}$ (Note. This is valid provided that $t < 1/2$).

$(1-2t)^{-1/2}$ is the moment-generating function for a gamma-distributed random variable with $\alpha = 1/2$ and $\beta = 2$.

Hence, Z^2 has a χ^2 distribution with $\nu = 1$ degree of freedom.

Exercise 6.40

Suppose that Y_1 and Y_2 are independent, standard Normal random variables. Find the probability distribution of $U = Y_1^2 + Y_2^2$.

Solution

$$
\begin{aligned}
M_U(t) &= E[e^{Ut}] = E[e^{(Y_1^2+Y_2^2)t}] \\
&= E[e^{Y_1^2 t} e^{Y_2^2 t}] \quad \text{(by independence)} \\
&= E[e^{Y_1^2 t}]E[e^{Y_2^2 t}] \\
&= M_{Y_1^2}(t) M_{Y_2^2}(t) \\
&= [(1-2t)^{-1/2}][(1-2t)^{-1/2}] = (1-2t)^{-2/2}.
\end{aligned}
$$

Because moment-generating functions are unique, U has a χ^2 distribution with 2 degrees of freedom.

Comment about last example

Note that $(1-2t)^{-2/2} = (1-2t)^{-1}$ which is the moment-generating function of an exponential random variable with parameter $\beta = 2$. Which is the right probability distribution? χ^2 with 2 df? Exponential with $\beta = 2$? Let us write the pdf for each of them.

Exponential pdf with $\beta = 2$.

$f(y) = \frac{1}{2}e^{-y/2}$, $0 < y < \infty$.

Chi-square pdf with $\nu = 2$.

$f(y) = \frac{y^{2/2-1}}{2^{2/2}\Gamma(2/2)}e^{-y/2} = \frac{1}{2}e^{-y/2}$, $0 < y < \infty$.

They are the same!

Example

Let Y_1 and Y_2 be independent, Normal random variables, each with mean μ and variance σ^2. Let a_1 and a_2 denote known constants. Find the density function of the linear combination $U = a_1 Y_1 + a_2 Y_2$.

Solution

The mgf for a Normal distribution with parameters μ and σ is $m(t) = e^{\mu t + \sigma^2 t^2/2}$.

$$\begin{aligned} M_U(t) &= E[e^{Ut}] = E[e^{(a_1 Y_1 + a_2 Y_2)t}] \\ &= E[e^{(a_1 Y_1)t} e^{(a_2 Y_2)t}] \text{ (by independence)} \\ &= E[e^{(a_1 Y_1)t}] E[e^{(a_2 Y_2)t}] \\ &= M_{Y_1}(a_1 t) M_{Y_2}(a_2 t) \\ &= [e^{\mu a_1 t + \sigma^2 (a_1 t)^2/2}][e^{\mu a_2 t + \sigma^2 (a_2 t)^2/2}] \\ &= e^{\mu t(a_1 + a_2) + \sigma^2 (a_1^2 + a_2^2) t^2/2} \end{aligned}$$

This is the mgf for a Normal variable with mean $\mu(a_1 + a_2)$ and variance $\sigma^2(a_1^2 + a_2^2)$.

Example

Let Y_1 and Y_2 be independent, Normal random variables, each with mean μ and variance σ^2. Find the density function of $\bar{Y} = \frac{Y_1+Y_2}{2}$.

Solution

From our previous example and making $a_1 = a_2 = \frac{1}{2}$, we have that $\bar{Y}$ has a Normal distribution with mean μ and variance $\sigma^2/2$.

Exercise 6.59

Show that if Y_1 has a χ^2 distribution with ν_1 degrees of freedom and Y_2 has a χ^2 distribution with ν_2 degrees of freedom, then $U = Y_1 + Y_2$ has a χ^2 distribution with $\nu_1 + \nu_2$ degrees of freedom, provided that Y_1 and Y_2 are independent.

Exercise 6.72 a.

Let Y_1 and Y_2 be independent and uniformly distributed over the interval $(0, 1)$. Find the probability density function of $U = min(Y_1, Y_2)$.

Solution

Let $U = min(Y_1, Y_2)$.
$F_U(u) = P(U \leq u) = 1 - P(U > u)$. Now, let us find $P(U > u)$.
$P(U > u) = P(min(Y_1, Y_2) > u) = [P(Y_1 > u)][P(Y_2 > u)]$
$P(U > u) = [1 - P(Y_1 \leq u)][1 - P(Y_2 \leq u)]$
$P(U > u) = [1 - u]^2$
Therefore, $F_U(u) = P(U \leq u) = 1 - [1 - u]^2$.
Finally, $f_U(u) = \frac{d}{du} F_U(u) = -2(1 - u)(-1) = 2(1 - u)$, $0 < u < 1$.

Exercise 6.73 a.

Let Y_1 and Y_2 be independent and uniformly distributed over the interval $(0, 1)$. Find the probability density function of $U_2 = max(Y_1, Y_2)$.

Solution

Let $U = max(Y_1, Y_2)$.
$F_U(u) = P(U \leq u) = P(max(Y_1, Y_2) \leq u)$
$= P(Y_1 \leq u)P(Y_2 \leq u) = (u)(u) = u^2$.
Therefore, $F_U(u) = u^2$.
Finally, $f_U(u) = \frac{d}{du}F_U(u) = 2u$, $0 < u < 1$.

Example

Let $Y_1, Y_2, ..., Y_n$ be independent, uniformly distributed random variables on the interval $[0, \theta]$. Find the pdf of $Y_{(n)}$.

Solution

Let $U = max(Y_1, Y_2, ..., Y_n)$.
$F_U(u) = P(U \leq u) = P(max(Y_1, Y_2, ..., Y_n) \leq u)$
$= P(Y_1 \leq u)P(Y_2 \leq u)...P(Y_n \leq u) = (u/\theta)(u/\theta)...(u/\theta)$.
Therefore, $F_U(u) = (u/\theta)^n$.
Finally, $f_U(u) = \frac{d}{du}F_U(u) = \frac{nu^{n-1}}{\theta^n}$, $0 \leq u \leq \theta$.

Example

The values $x_1 = 0.62$, $x_2 = 0.98$, $x_3 = 0.31$, $x_4 = 0.81$, and $x_5 = 0.53$ are the $n = 5$ observed values of five independent trials of an experiment with pdf $f(x) = 2x, \ 0 < x < 1$. The observed order statistics are
$y_1 = 0.31 < y_2 = 0.53 < y_3 = 0.62 < y_4 = 0.81 < y_5 = 0.98$.
Recall that the middle observation in the ordered arrangement, here $y_3 = 0.62$ is called the sample median and the difference of the largest and the smallest here
$y_5 - y_1 = 0.98 - 0.31 = 0.67$,
is called the sample range.

If $X_1, X_2, ..., X_n$ are observations of a random sample of size n from a continuous-type distribution, we let the random variables
$Y_1 < Y_2 < ... < Y_n$
denote the order statistics of that sample. That is,
$Y_1 =$ smallest of $X_1, X_2, ..., X_n$,
$Y_2 =$ second smallest of $X_1, X_2, ..., X_n$,
. . .

$Y_n =$ largest of $X_1, X_2, ..., X_n$.

Example

Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the order statistics of a random sample X_1, X_2, X_3, X_4, X_5 of size $n = 5$ from the distribution with pdf $f(x) = 2x$, $0 < x < 1$. Consider $P(Y_4 \leq 1/2)$.

Example (cont.)

For the event $Y_4 \leq 1/2$ to occur, at least four of the random variables X_1, X_2, X_3, X_4, X_5 must be less than $1/2$.
Thus if the event $X_i \leq 1/2$, $i = 1, 2, ..., 5$, is called "success" we must have at least four successes in the five mutually independent trials, each of which has probability of success
$P\left(X_i \leq \frac{1}{2}\right) = \int_0^{1/2} 2x dx = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$
Thus,
$P\left(y_4 \leq \frac{1}{2}\right) = \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^5 = 0.0156$

Example (cont.)

In general, if $0 < y < 1$, then the distribution function of Y_4 is

$$G(y) = P(Y_4 \leq y) = \binom{5}{4} \left(y^2\right)^4 \left(1 - y^2\right) + \left(y^2\right)^5$$

since this represents the probability of at least four "successes" in five independent trials, each of which has probability of success

$$P(X_i \leq y) = \int_0^y 2x dx = y^2.$$

Example (cont.)

The pdf of Y_4 is therefore, for $0 < y < 1$,

$$g(y) = G'(y) = \binom{5}{4} 4(y^2)^3(2y)(1-y^2) + \binom{5}{4}(y^2)^4(-2y) + 5(y^2)^4(2y)$$

$$g(y) = \frac{5!}{3!1!}(y^2)^3(1-y^2)(2y), \quad 0 < y < 1.$$

Note that in this example, the cumulative distribution function of each X is $F_X(x) = x^2$ when $0 < x < 1$. Thus

$$g(y) = \frac{5!}{3!1!}[F_X(y)]^3(1 - F_X(y)]f(y), \quad 0 < y < 1.$$

Theorem 6.5

Let $Y_1, Y_2, ..., Y_n$ be independent identically distributed continuous random variables with common distribution function $F(y)$ and common density function $f(y)$. If $Y_{(k)}$ denotes the kth-order statistic, then the density function of $Y_{(k)}$ is given by

$$g_{(k)}(y_k) = \frac{n!}{(k-1)!(n-k)}[F(y_k)]^{k-1}[1-F(y_k)]^{n-k}f(y_k),$$

$\infty < y_k < \infty$

Bivariate Transformation Method

Let X_1 and X_2 be jointly continuous random variables with joint probability density function $f_{X_1,\ X_2}$. It is sometimes necessary to obtain the joint distribution of the random variables Y_1 and Y_2, which arise as functions of X_1 and X_2. Specifically, suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ for some functions g_1 and g_2. Assume that the functions g_1 and g_2 satisfy the following conditions:
1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2 with solutions given by, say, $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.

Bivariate Transformation Method

2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the following 2×2 determinant

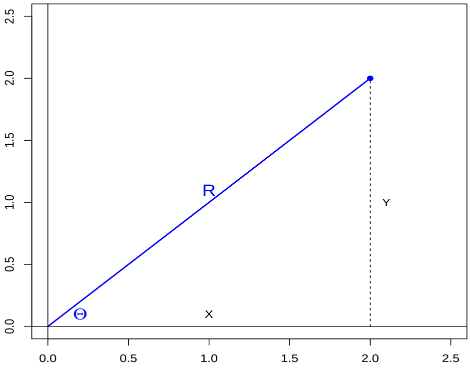
$$J(x_1, x_2) = det \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} \neq 0$$

at all points (x_1, x_2).
Under these two conditions it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by
$f_{Y_1,\ Y_2}(y_1,\ y_2) = f_{X_1,\ X_2}(x_1,\ x_2)|J(x_1, x_2)|^{-1}$,
where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$ and $|J(x_1, x_2)|$ is the absolute value of the Jacobian.

Example

Let $(X, \ Y)$ denote a random point in the plane and assume that the rectangular coordinates X and Y are independent standard random Normal random variables. We are interested in the joint distribution of R and Θ, the polar coordinate representation of this point (see Figure below).

Figure



Example

Letting $r = g_1(x,\ y) = \sqrt{x^2 + y^2}$ and $\theta = g_2(x,\ y) = tan^{-1}\left(\frac{y}{x}\right)$, we see that

$$\frac{\partial g_1}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial g_1}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial g_2}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial g_2}{\partial y} = \frac{1}{x[1+(y/x)^2]} = \frac{x}{x^2+y^2}$$

Example

Hence
$J = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$.
As the joint density function of X and Y is

$$f_{X,\ Y}(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

we see that the joint density function of $R = \sqrt{x^2 + y^2}$,
$\Theta = tan^{-1}(y/x)$, is given by

$$f_{R,\ \Theta}(r,\ \theta) = \frac{1}{2\pi} re^{-r^2/2} \quad 0 < \theta < 2\pi,\ \ 0 < r < \infty.$$

Example

As this joint density factors into the marginal densities for R and Θ, we obtain that R and Θ are independent random variables, with Θ being uniformly distributed over $(0, 2\pi)$ and R having the Rayleigh distribution with density

$$f_R(r) = re^{-r^2/2} \quad 0 < r < \infty.$$

Example

If we wanted the joint distribution of R^2 and Θ, then, as the transformation $d = h_1(x,\ y) = x^2 + y^2$ and $\theta = h_2(x,\ y) = tan^{-1}(y/x)$ has a Jacobian

$$J = det \begin{bmatrix} 2x & 2y \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} = 2$$

we see that

$$f_{D,\ \Theta}(d,\ \theta) = \frac{1}{2} e^{-d/2} \frac{1}{2\pi} \quad 0 < d < \infty,\ \ 0 < \theta < 2\pi.$$

Therefore, R^2 and Θ are independent, with R^2 having an exponential distribution with parameter $\beta = 2$.

We would like to show why $\Gamma(1/2) = \sqrt{\pi}$. First, we integrate a standard Normal random variable over its entire domain.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$$

Notice that the integrand above is symmetric around 0. Thus,

$$\int_{0}^{\infty} e^{-z^2/2} dz = \frac{\sqrt{2\pi}}{2} = \sqrt{\pi/2}$$

Now, let $w = \frac{z^2}{2}$, which implies that $dz = (2w)^{-1/2} dw$. Then

$$\int_{0}^{\infty} e^{-z^2/2} dz = \int_{0}^{\infty} (2w)^{-1/2} e^{-w} dw = \sqrt{\pi/2}$$

Our last equation is equivalent to

$$\int_0^\infty (w)^{-1/2} e^{-w} dw = \sqrt{\pi}$$

Next, we multiply the last integral by a "one"

$$\Gamma(1/2) \int_0^\infty \frac{1}{\Gamma(1/2)} (w)^{-1/2} e^{-w} dw = \sqrt{\pi}$$

We notice that the last integral equals one (we are integrating a Gamma distribution over its entire domain). Therefore

$$\Gamma(1/2)(1) = \sqrt{\pi}$$

Homework?

Let X_1 and X_2 be jointly continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2}.