Testing Hypotheses

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Suppose we have a coin that either is honest or is a coin that has been weighted so that when tossed its probability of coming up heads is 0.6. We wish to test whether the coin is honest or is the weighted coin by tossing it three times and observing the number of heads that is obtained. Our sample here is the triple of numbers (x_1, x_2, x_3), where $x_i = 1$ or 0 corresponding to whether a head or a tail was obtained on the ith toss.

We may treat this as a problem of testing the hypothesis $H_0: \ \theta = 0.5$ vs $H_a: \ \theta = 0.6$, where X is a Bernoulli random variable with parameter θ and from which a random sample of size 3 has been taken. Since there are only two possible actions that can be taken in a testing problem, namely accept H_0 or accept H_a, a decision function (also known as tests statistic) $W = W(x_1, x_2, x_3)$ must separate 3 dimensional space into two parts. Let A_0 denote the part that is associated with accepting H_0, and A_a the remaining part associated with accepting H_a ($A_a =$ Rejection Region or RR). This means that if a random sample of X yields a point (x_1, x_2, x_3) that lies in A_0, we accept the hypothesis $H_0 : \theta = 0.5$ whereas if it lies in RR, we reject H_0 and accept the alternative hypothesis $H_a : \theta = \theta_a$.

Definition

A **type I error** is made if H_0 is rejected when H_0 is true. The probability of a type I error is denoted by α. The value of α is called the level of the test.

A **type II error** is made if H_0 is accepted when H_a is true. The probability of a type II error is denoted by β.

Problem

Suppose you are testing $H_0: \ p = 1/2$ against $H_1 : p = 2/3$ for a Binomial variable X with $n = 3$. What values of X would you assign to the rejection region (RR) if you wish to have $\alpha \leq 1/8$ and you wish to minimize β corresponding to the value of α selected?

Solution

(We can find the pmf of X in our table).

x	0	1	2	3
$f(x \mid p = 1/2)$	$1/8$	$3/8$	$3/8$	$1/8$
$f(x \mid p = 2/3)$	$1/27$	$6/27$	$12/27$	$8/27$

Solution

First, recall α's definition

$\alpha = P$(test statistic is in RR when H_0 is true)

$\alpha = P(X \in RR | p = 1/2)$

Solution

Proposal one: $RR = \{X = 0\}$ (clearly, this rejection region has an $\alpha = 1/8$)
$\beta = P(\text{accepting } H_0 \text{ when } H_a \text{ is true})$
$\beta = P(\text{value of the test statistic is not in RR when } H_a \text{ is true})$
$\beta = P(X = 1 \text{ or } X = 2 \text{ or } X = 3 | p = 2/3) = 26/27$

Solution

Note that
$\beta = P(X = 1 \text{ or } X = 2 \text{ or } X = 3 | p = 2/3)$
$1 - \beta = P(X = 0 | p = 2/3)$
$1 - \beta = 1/27$
(this quantity, $1 - \beta$, will be called the power of the test).

Solution

Proposal two: $RR = \{X = 3\}$ (clearly, this rejection region has an $\alpha = 1/8$)
$\beta = P(\text{accepting } H_0 \text{ when } H_a \text{ is true})$
$\beta = P(\text{value of the test statistic is not in RR when } H_a \text{ is true})$
$\beta = P(X = 0 \textit{ or } X = 1 \textit{ or } X = 2 | p = 2/3) = 19/27$

Solution

Note that
$\beta = P(X = 0 \text{ or } X = 1 \text{ or } X = 2 | p = 2/3)$
$1 - \beta = P(X = 3 | p = 2/3)$
$1 - \beta = 8/27$
(this quantity, $1 - \beta$, will be called the power of the test).

Hence choose second proposal ($RR = \{X = 3\}$) because the size of its type II error is smaller.

Solution

Also note that

$\alpha_{proposal\ 1} = \alpha_{proposal\ 2}$.

On the other hand,

$\frac{1}{27} < \frac{8}{27}$

Power of proposal 1 $<$ Power of proposal 2.

Which implies that proposal 2 is "more powerful" than proposal 1.

Definition

Suppose that W is the test statistic and RR is the rejection region for a test of a hypothesis involving the value of a parameter θ. Then the power of the test, denoted by power(θ), is the probability that the test will lead to rejection of H_0 when the actual parameter value is θ. That is,
power(θ) = P(W in RR when the parameter value is θ)

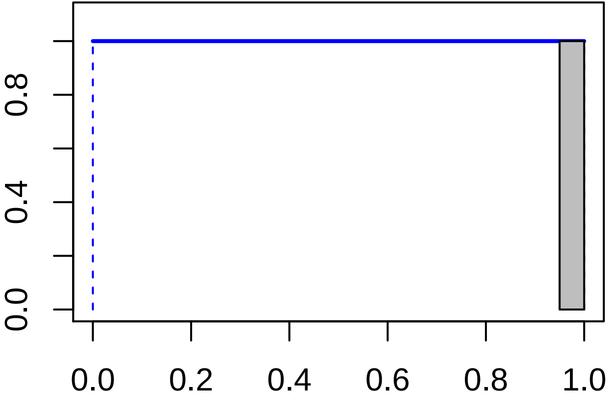
Exercise 10.5

Let Y_1 and Y_2 be independent and identically distributed with a uniform distribution over the interval $(\theta, \theta + 1)$. For testing $H_0 : \theta = 0$ vs $H_a : \theta > 0$, we have two competing tests:
Test 1: Reject H_0 if $Y_1 > 0.95$
Test 2: Reject H_0 if $Y_1 + Y_2 > C$
Find the value of C so that test 2 has the same value for α as test 1.

Solution

Recall that α = P(rejecting H_0 when H_0 is true).
Test 1.
$\alpha = P(Y_1 > 0.95 \text{ when } \theta = 0) = 0.05$
(See figure)

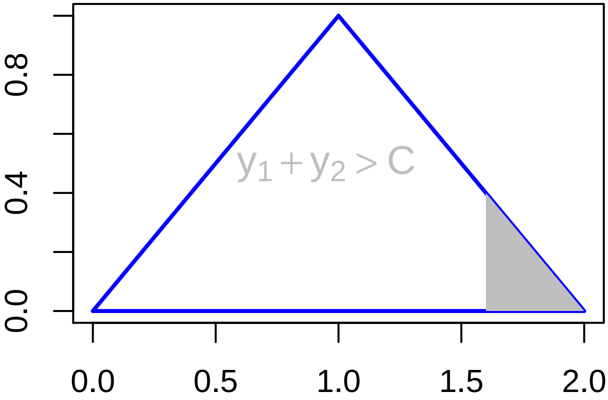
Figure



Solution

Test 2
$\alpha = 0.05 = P(Y_1 + Y_2 > C$ when H_0 is true)
Recall that when Y_1 and Y_2 have a uniform distribution over $(0, 1)$ then the pdf of $Y_1 + Y_2$ is given by the function shown below (see example 6.3, it was one of the first things we did together this semester).

Figure



Solution

Area of triangle $= \frac{bh}{2}$
$0.05 = \frac{(2-C)(2-C)}{2}$
$0.05 = \frac{(2-C)^2}{2}$
(solving for C)
$C \approx 1.6837$

The Neyman-Pearson Lemma

Suppose that we wish to test the simple null hypothesis $H_0 : \theta = \theta_0$ versus the simple alternative hypothesis $H_a : \theta_a$, based on a random sample $Y_1, \; Y_2, ..., Y_n$ from a distribution with parameter θ. Let $L(\theta)$ denote the likelihood of the sample when the value of the parameter is θ. Then, for a given α, the test that maximizes the power at θ_a has a rejection region, RR, determined by

$$\frac{L(\theta_0)}{L(\theta_a)} < k.$$

The value of k is chosen so that the test has the desired value for α. Such a test is a most powerful α-level test for H_0 versus H_a.

Example

Suppose that Y represents a single observation from a population with probability density function given by

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1, \\ 0, & \textit{elsewhere}. \end{cases}$$

Find the most powerful test with significance level $\alpha = 0.05$ to test $H_0 : \theta = 2$ versus $H_a : \theta = 1$.

Solution

$\frac{L(\theta_0)}{L(\theta_a)} = \frac{f(y|\theta_0)}{f(y|\theta_a)} = \frac{2y^{2-1}}{1} = 2y$ for $0 < y < 1$, and the form of the rejection region for the most powerful test is

$$2y < k.$$

Thus, RR is $\{y < k/2\}$ or $\{y < k^*\}$.

Solution

Recalling that $\alpha = 0.05$ and its definition, we have that
$0.05 = \text{P}(Y \text{ in RR when } H_0 \text{ is true}) = \text{P}(Y \text{ in RR when } \theta = 2)$
$= P(Y < k^*) \text{ when } \theta = 2)$
$= \int_0^{k^*} 2y dy.$
Therefore, $(k^*)^2 = 0.05$, and the rejection region of the most powerful test is

$$RR = \{y < \sqrt{0.05} = 0.2236\}.$$

Example

Let $Y_1, Y_2, X_3, ..., Y_n$ be a random sample from the Normal distribution $N(\mu, \sigma^2 = 36)$. We shall find the best critical region (or most powerful test) for testing the simple hypothesis
$H_0 : \mu_0 = 50$ vs $H_a : \mu = 55$.
(In this example, $\mu_0 = 50$ and $\mu_a = 55$).

Solution

From our table, we have that

$$f(y|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}.$$

Solution

Applying N-P Lemma, we have that

$$\frac{L(\mu_0)}{L(\mu_a)} = \frac{(72\pi)^{-n/2} exp\left(-\frac{1}{72}\sum(y_i - 50)^2\right)}{(72\pi)^{-n/2} exp\left(-\frac{1}{72}\sum(y_i - 55)^2\right)}$$

$$\frac{L(\mu_0)}{L(\mu_a)} = exp\left\{-\frac{1}{72}\left[\sum(y_i - 50)^2 - \sum(y_i - 55)^2\right]\right\}$$

Solution

Let us "play" with the exponent, so we can simplify the last expression

$$\begin{aligned}\sum(y_i - 50)^2 - \sum(y_i - 55)^2 \\ &= \sum[y_i^2 - 100y_i + 2500] - \sum[y_i^2 - 110y_i + 3025] \\ &= 10\sum y_i - 525n.\end{aligned}$$

Solution

Thus,

$$\frac{L(\mu_0)}{L(\mu_a)} = exp\left\{-\frac{1}{72}\left[10\sum y_i - 525n\right]\right\} < k.$$

Now, let us find an equivalent RR that is "easier" to deal with

$\left\{-\frac{1}{72}\left[10\sum y_i - 525n\right]\right\} < ln(k)$

$\sum y_i > \frac{525n - 72ln(k)}{10}$ (dividing by n on both sides)

$\bar{y} > k^*$

Solution

Thus $\frac{L(\mu_0)}{L(\mu_a)} < k$ is equivalent to $RR = \{\bar{y} > k^*\}$. A best critical region is, according to Neyman-Pearson lemma, $\{\bar{y} > k^*\}$ where k^* is selected so that the size of the critical region (or rejection region) is α.

Solution

Say $n = 16$ with $\alpha = 0.05$

$$\alpha = 0.05 = P(\bar{Y} > k^*|\mu = 50)$$
$$= P\left(\frac{\bar{Y}-50}{6/4} > \frac{k^*-50}{6/4}\right)$$
$$= P(Z > 1.645).$$

Solving for k^* (from $\frac{4(k^*-50)}{6} = 1.645$), we have that $k^* = 52.4675$.

Finally!! $RR = \{\bar{y} > 52.4675\}$

Example

Suppose that $Y_1, Y_2, ..., Y_n$ constitute a random sample from a Normal distribution with unknown mean μ and **known** variance σ^2. We wish to test $H_0 : \mu = \mu_0$ against $H_a : \mu > \mu_0$ for a specified constant μ_0. Find the **uniformly most powerful test** with significance level α.

Let us "recycle" our work from the previous problem. That is, we will start by finding the most powerful α-level test of $H_0 : \mu = \mu_0$ against $H_a : \mu = \mu_a$ (where μ_a is a fixed value such that $\mu_a > \mu_0$).

Solution

Applying N-P Lemma, we have that

$$\frac{L(\mu_0)}{L(\mu_a)} = \frac{(2\pi\sigma^2)^{-n/2} exp\left(-\frac{1}{2\sigma^2}\sum(y_i - \mu_0)^2\right)}{(2\pi\sigma^2)^{-n/2} exp\left(-\frac{1}{2\sigma^2}\sum(y_i - \mu_a)^2\right)}$$

$$\frac{L(\mu_0)}{L(\mu_a)} = exp\left\{-\frac{1}{2\sigma^2}\left[\sum(y_i - \mu_0)^2 - \sum(y_i - \mu_a)^2\right]\right\}$$

Solution

Let us "play" with the exponent, so we can simplify the last expression

$$\sum(y_i-\mu_0)^2-\sum(y_i-\mu_a)^2$$
$$=\sum(y_i^2-2\mu_0 y_i+\mu_0^2)-\sum(y_i^2-2\mu_a y_i+\mu_a^2)$$
$$=-2\mu_0\sum y_i+n\mu_0^2+2\mu_a\sum y_i-n\mu_a^2$$
$$=-2n\mu_0\bar{y}+n\mu_0^2+2n\mu_a\bar{y}-n\mu_a^2$$

Solution

Thus,

$$\frac{L(\mu_0)}{L(\mu_a)} = exp\left\{-\frac{1}{2\sigma^2}[(2n\mu_a - 2n\mu_0)\bar{y} + n(\mu_0^2 - \mu_a^2)]\right\} < k.$$

Now, let us find an equivalent RR that is "easier" to deal with
$\{-\frac{1}{2\sigma^2}[(2n\mu_a - 2n\mu_0)\bar{y} + n(\mu_0^2 - \mu_a^2)]\} < ln(k)$
$2n(\mu_a - \mu_0)\bar{y} + n(\mu_0^2 - \mu_a^2) > -2\sigma^2 ln(k)$
$2n(\mu_a - \mu_0)\bar{y} > -2\sigma^2 ln(k) - n(\mu_0^2 - \mu_a^2)$
(dividing by $2n(\mu_a - \mu_0)$ on both sides and noting that this quantity is positive, for any μ_a such that $\mu_a > \mu_0$)
$\bar{y} > k^*$

Solution

Thus $\frac{L(\mu_0)}{L(\mu_a)} < k$ is equivalent to $RR = \{\bar{y} > k^*\}$. Therefore, the most powerful test of $H_0 : \mu = \mu_0$ vs $H_a : \mu = \mu_a$ has a rejection region given by $\{\bar{y} > k^*\}$ where k^* is selected so that the size of the rejection region is α.

Solution

$$\alpha = P(\bar{Y} > k^*|\mu = \mu_0)$$
$$= P\left(\frac{\bar{Y}-\mu_0}{\sigma/\sqrt{n}} > \frac{k^*-\mu_0}{\sigma/\sqrt{n}}\right)$$
$$= P(Z > z_\alpha).$$

Solving for k^* (from $\frac{\sqrt{n}(k^*-\mu_0)}{\sigma} = z_\alpha$), we have that $k^* = \mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)$.

Finally!! $RR = \{\bar{y} > \mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)\}$

Solution

We now observe that neither the test statistic ($\bar{y}$) nor the rejection region for this α-level test depends on the particular value assigned to μ_a. **That is, for any value of μ_a that satisfies the condition $\mu_a > \mu_0$, we obtain exactly the same RR***. Thus, the test with the RR that we found above has the largest possible value for power(μ_a) for every $\mu_a > \mu_0$. It is the **uniformly most powerful test** for $H_0 : \mu_0$ vs $H_a : \mu > \mu_0$.

Exercise 10.95

Suppose that we have a random sample of four observations from the density function

$$f(y|\theta) = \begin{cases} \frac{1}{2\theta^3} y^2 e^{-y/\theta}, & y > 0, \\ 0, & \textit{elsewhere}. \end{cases}$$

a. Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ vs $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$.
b. Is the test given in part (a) uniformly most powerful for the alternative $\theta > \theta_0$?

Likelihood Ratio Tests

We let Ω denote the total parameter space, that is, the set of all possible values of the parameter θ given by either H_0 or H_a. These hypotheses will be stated as follows:

$H_0 : \theta \in \omega, \qquad H_a : \theta \in \omega^{'}$,

where ω is a subset of Ω and $\omega^{'}$ is the complement of ω with respect to Ω.

Definition

The Likelihood ratio is the quotient

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})},$$

where $L(\hat{\omega})$ is the maximum of the likelihood function with respect to θ when $\theta \in \omega$ and $L(\hat{\Omega})$ is the maximum of the likelihood function with respect to θ when $\theta \in \Omega$.

Definition

To test $H_0 : \theta \in \omega$ against $H_a : \theta \in \omega'$, the **Rejection Region (RR)** (a.k.a. critical region) for the likelihood ratio test is the set of points in the sample space for which

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \leq k$$

where $0 < k < 1$ and k is selected so that the test has a desired significance level α.

Example

We shall test the hypothesis
$H_0 : \mu = 162$ vs $H_a : \mu \neq 162$
for a Normal variable with known $\sigma^2 = 5$ based on a random sample of size n and $\alpha = 0.05$.

Step 0

Thus $\omega = \{162\}$ and $\Omega = \{\mu : -\infty < \mu < \infty\}$.

Step 1. Finding Likelihood

$$
\begin{aligned}
L(y_1, y_2, ..., y_n|\mu) &= f(y_1|\mu)...f(y_n|\mu) \\
&= \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(y_1-\mu)^2} \cdots \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(y_n-\mu)^2} \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} = L(\mu)
\end{aligned}
$$

Step 2. Finding $L(\hat{\omega})$

When H_0 is true, μ can take on only one value. Thus $L(\hat{\omega}) = L(162)$.

$$L(\hat{\omega}) = \left(\frac{1}{10\pi}\right)^{n/2} e^{-\frac{1}{10}\sum_{i=1}^{n}(y_i - 162)^2}$$

Step 3. Finding $L(\hat{\Omega})$

To find $L(\hat{\Omega})$, we must find the value of μ that maximizes $L(\mu)$ (recall that it is easier to maximize $lnL(\mu)$).

$$
\begin{aligned}
lnL(\mu) &= \frac{n}{2} ln\left(\frac{1}{10\pi}\right) - \frac{1}{10}\sum_{i=1}^{n}(y_i-\mu)^2 \\
&= -\frac{n}{2} ln(10\pi) - \frac{1}{10}\sum_{i=1}^{n}(y_i-\mu)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{dlnL(\mu)}{d\mu} &= -\frac{1}{10}\sum_{i=1}^{n} 2(y_i-\mu)(-1) \\
&= \frac{1}{5}\sum_{i=1}^{n}(y_i-\mu)
\end{aligned}
$$

(now, we have to set it equal to zero and solve for μ)

Step 3 (cont.)

$\sum_{i=1}^{n} y_i - n\mu = 0$
$\sum_{i=1}^{n} y_i = n\mu$
$\hat{\mu}_{MLE} = \bar{y}$
(check that it is a max...)
Thus $L(\hat{\Omega}) = L(\bar{y})$.

Step 4. Finding λ

$$\lambda = \frac{L(162)}{L(\bar{y})} = \frac{\left(\frac{1}{10\pi}\right)^{n/2} e^{-\frac{1}{10}\sum_{i=1}^{n}(y_i - 162)^2}}{\left(\frac{1}{10\pi}\right)^{n/2} e^{-\frac{1}{10}\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

(Now, let us try and simplify λ)

$\sum_{i=1}^{n}(y_i - 162)^2 = \sum_{i=1}^{n}[(y_i - \bar{y}) + (\bar{y} - 162)]^2$

$= \sum_{i=1}^{n}(y_i - \bar{y})^2 + 2(\bar{y} - 162)\sum_{i=1}^{n}(y_i - \bar{y}) + n(\bar{y} - 162)^2$

(note that $\sum_{i=1}^{n}(y_i - \bar{y}) = 0$)

$= \sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - 162)^2$

Step 4 (cont.)

$$\lambda = \frac{exp\left[-\frac{1}{10}\sum_{i=1}^{n}(y_i - \bar{y})^2 - \frac{n}{10}(\bar{y} - 162)^2\right]}{exp\left[-\frac{1}{10}\sum_{i=1}^{n}(y_i - \bar{y})^2\right]}$$

$$\lambda = e^{-\frac{n}{10}(\bar{y}-162)^2}.$$

Step 5. Finding RR

The Rejection Region (RR) is given by

$$exp\{-\frac{n}{10}(\bar{y}-162)^2\} \leq k$$

(which is equivalent to)

$$(\bar{y}-162)^2 \geq -\frac{10}{n} ln(k)$$

Step 5 (cont.)

Note that when $H_0 : \mu = 162$ is true

$$\frac{\bar{y} - 162}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{y} - 162}{\frac{\sqrt{5}}{\sqrt{n}}} = Z \quad i.e.\ a\ N(0,1).$$

Therefore,

$$\left[\frac{\bar{y} - 162}{\frac{\sqrt{5}}{\sqrt{n}}}\right]^2 = \chi^2(1) \geq k^*$$

Step 5 (cont.)

From Table 6 and using $\alpha = 0.05$, $k^* \approx 3.84146$. Thus the Rejection Region is:

$$RR = \{\bar{y} : \left[\frac{\bar{y} - 162}{\frac{\sqrt{5}}{\sqrt{n}}} \right]^2 \geq 3.84146\}$$

or

$$RR = \{\bar{y} : \frac{n(\bar{y} - 162)^2}{5} \geq 3.84146\}$$

Theorem 10.2

Let $Y_1, Y_2, ..., Y_n$ have joint likelihood function $L(\Theta)$. Let r_0 denote the number of free parameters that are specified by $H_0 : \Theta \in \Omega_0$ and let r denote the number of free parameters specified by the statement $\Omega \in \Omega$. Then, for large n, $-2ln(\lambda)$ has approximately a χ^2 distribution with $r - r_0$ df.

Example

As an illustration of how the asymptotic distribution performs on a familiar problem, let us apply it to testing
$H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$
for a Normal variable with **known** σ^2 based on a random sample of size n and $\alpha = 0.05$.

Example (cont.)

Hence the likelihood functions are

$$L(\mu_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu_0)^2}$$

and

$$L(\bar{y}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\bar{y})^2}$$

Example (cont.)

Performing some algebraic simplifications on the likelihood ratio will produce the value

$$\lambda = exp\left[-\frac{n(\bar{y}-\mu_0)^2}{2\sigma^2}\right].$$

Hence,

$$-2ln(\lambda) = \frac{n(\bar{y}-\mu_0)^2}{\sigma^2} = \left(\frac{\bar{y}-\mu_0}{\frac{\sigma}{\sqrt{n}}}\right)^2$$

Example (cont.)

Since $\left(\frac{\bar{y}-\mu_0}{\frac{\sigma}{\sqrt{n}}}\right)^2$ is a Standard Normal variable when H_0 is true, we know that $\left(\frac{\bar{y}-\mu_0}{\frac{\sigma}{\sqrt{n}}}\right)^2$ possesses an exact chi-square distribution with one degree of freedom. Thus, the approximation here for large n happens to be exact!

Another example

Suppose now that the random sample $Y_1, Y_2, ..., Y_n$ arises from the Normal population $N(\mu, \sigma^2)$ where both μ and σ^2 are **unknown**. Let us consider the likelihood ratio test of the null hypothesis $H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$.

Step 0

For this test
$\omega = \{(\mu, \sigma^2) : \mu = \mu_0, 0 < \sigma^2 < \infty\}$ and
$\Omega = \{(\mu, \sigma^2) : -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$

Step 1. Finding Likelihood

$$L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

If $(\mu, \sigma^2) \in \omega$, the maximum likelihood estimates are $\hat{\mu} = \mu_0$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu_0)^2$ (Remember?)

$$L(\hat{\omega}) = \left(\frac{1}{\frac{2\pi}{n}\sum(y_i - \mu_0)^2}\right)^{n/2} exp\left(-\frac{\sum(y_i - \mu_0)^2}{\frac{2}{n}\sum(y_i - \mu_0)^2}\right)$$

$$L(\hat{\omega}) = \left(\frac{ne^{-1}}{2\pi\sum(y_i - \mu_0)^2}\right)^{n/2}$$

Step 3. Finding $L(\hat{\Omega})$

If $(\mu, \sigma^2) \in \Omega$, the maximum likelihood estimates are $\hat{\mu} = \bar{y}$ and $\hat{\sigma}^2 = \frac{\sum(y_i - \bar{y})^2}{n}$

$$L(\hat{\Omega}) = \left(\frac{1}{\frac{2\pi}{n} \sum(y_i - \bar{y})^2} \right)^{n/2} exp \left(- \frac{\sum(y_i - \bar{y})^2}{\frac{2}{n} \sum(y_i - \bar{y})^2} \right)$$

$$L(\hat{\Omega}) = \left(\frac{ne^{-1}}{2\pi \sum(y_i - \bar{y})^2} \right)^{n/2}$$

Step 4. Finding λ

$$\lambda = \frac{\left(\frac{ne^{-1}}{2\pi\sum(y_i-\mu_0)^2}\right)^{n/2}}{\left(\frac{ne^{-1}}{2\pi\sum(y_i-\bar{y})^2}\right)^{n/2}}$$

Note that

$$\sum (y_i - \mu_0)^2 = \sum (y_i - \bar{y})^2 + n(\bar{y} - \mu_0)^2$$

Step 4. Finding λ

$$\lambda = \left(\frac{\sum (y_i - \bar{y})^2}{\sum (y_i - \bar{y})^2 + n(\bar{y} - \mu_0)^2} \right)^{n/2} \leq k$$

Step 5. Finding RR

We know that the Rejection Region (RR) is given by

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \leq k$$

which is equivalent to

$$\frac{1}{k} \leq \frac{L(\hat{\Omega})}{L(\hat{\omega})} = \frac{1}{\lambda}.$$

Step 5. Finding RR

$$\frac{1}{\lambda} = \frac{\sum(y_i - \bar{y})^2 + n(\bar{y} - \mu_0)^2}{\sum(y_i - \bar{y})^2}$$

$$\frac{1}{\lambda} = 1 + \frac{n(\bar{y} - \mu_0)^2}{\sum(y_i - \bar{y})^2} \geq \frac{1}{k}$$

Step 5. Finding RR

$$\frac{n(\bar{y}-\mu_0)^2}{\sum(y_i-\bar{y})^2} \geq \frac{1}{k} - 1$$

$$\frac{n(n-1)(\bar{y}-\mu_0)^2}{\sum(y_i-\bar{y})^2} \geq \left(\frac{1}{k} - 1\right)(n-1)$$

$$\frac{\frac{n\sigma^2}{\sigma^2}(n-1)(\bar{y}-\mu_0)^2}{\sum(y_i-\bar{y})^2} \geq C$$

Step 5. Finding RR

$$\frac{\frac{n(\bar{y}-\mu_0)^2}{\sigma^2}}{\frac{\sum(y_i-\bar{y})^2}{\sigma^2(n-1)}} \geq C$$

Step 5. Finding RR

When H_0 is true, $\sqrt{n}\frac{(\bar{y}-\mu_0)}{\sigma}$ is N(0,1) and

$\frac{\sum(y_i-\bar{y})^2}{\sigma^2} = \frac{\frac{(n-1)\sum(y_i-\bar{y})^2}{n-1}}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}$
has an independent chi-square distribution $\chi^2(n-1)$.

Step 5. Finding RR

Hence, under H_0

$$T = \frac{\sqrt{n}\frac{(\bar{y}-\mu_0)}{\sigma}}{\sqrt{\frac{1}{\sigma^2}\frac{\sum(y_i-\bar{y})^2}{n-1}}}$$

T has a t distribution with $n-1$ df.
In accordance with the likelihood ratio test criterion, H_0 is rejected if $T^2 \geq C$. That is, we reject $H_0 : \mu = \mu_0$ if the observed $|T| \geq t_{\alpha/2}(n-1)$.