STA258H5

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HYPOTHESIS TESTS AND CONFIDENCE INTERVALS FOR MEANS AND PROPORTIONS

The one-sample t test

Draw an SRS of size n from a large population having unknown mean μ. To *test the hypothesis* $H_0 : \mu = \mu_0$, compute the *one-sample t statistic*

$$t^* = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

In terms of a variable T having the $t(n-1)$ distribution, the P-value for a test of H_0 against
$H_a : \mu > \mu_0$ is $P(T \geq t^*)$.
$H_a : \mu < \mu_0$ is $P(T \leq t^*)$.
$H_a : \mu \neq \mu_0$ is $2P(T \geq |t^*|)$.
These P-values are exact if the population distribution is Normal and are approximately correct for large n in other cases.

Is it significant?

The one-sample t statistic for testing
$H_0 : \mu = 0$
$H_a : \mu > 0$
from a sample of $n = 20$ observations has the value $t^* = 1.84$.
a) What are the degrees of freedom for this statistic?
b) Give the two critical values t from Table that bracket t^*. What are the one-sided P-values for these two entries?
c) Is the value $t^* = 1.84$ significant at the 5% level? Is it significant at the 1% level?
d) (Optional) If you have access to suitable technology, give the exact one-sided P-value for $t^* = 1.84$?

Solution

a) df = 20 -1 = 19.
b) $t^* = 1.84$ is bracketed by $t = 1.729$ (with right-tail probability 0.05) and $t = 2.093$ (with right-tail probability 0.025). Hence, because this is a one-sided significance test, $0.025 <$ P-value < 0.05.
c) This test is significant at the 5% level because the P-value < 0.05. It is not significant at the 1% level because the P-value > 0.01.

Solution d)

```
1 - pt(1.84,df=19);

## [1] 0.04072234

# pt gives you the area to the left of 1.84
# for a T distribution with df =19;
```

Is it significant?

The one-sample t statistic from a sample of $n = 15$ observations for the two-sided test of
$H_0 : \mu = 64$
$H_a : \mu \neq 64$
has the value $t^* = 2.12$.
a) What are the degrees of freedom for t^*?
b) Locate the two-critical values t from Table that bracket t^*. What are the two-sided P-values for these two entries?
c) is the value $t^* = 2.12$ statistically significant at the 10% level? At the 5% level?
d) (Optional) If you have access to suitable technology, give the exact two-sided P-value for $t^* = 2.12$.

Solution

a) df = 15 -1 = 14.
b) $t^* = 2.12$ is bracketed by $t = 1.761$ (with two-tail probability 0.10) and $t = 2.145$ (with two-tail probability 0.05). Hence, because this is a two-sided significance test, $0.05 <$ P-value < 0.10.
c) This test is significant at the 10% level because the P-value < 0.10. It is not significant at the 5% level because the P-value > 0.05.

Solution d)

```
2*(1 - pt(2.12,df=14));

## [1] 0.05235683

# pt gives you the area to the left of 2.12
# for a T distribution with df =12;
```

Example

$H_0 : \mu = 12$
$H_a : \mu > 12$
A sample of 25 provided a sample mean $\bar{x} = 14$ and a sample standard deviation $s = 4.32$.

a. Compute the value of the test statistic.
b. Use the t distribution table to compute a range for the p-value.
c. At $\alpha = 0.05$, what is your conclusion?

Solution

a. $t_* = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{14-12}{4.32/\sqrt{25}} = 2.31$

b. Degrees of freedom $= n - 1 = 24$.

P-value $= P(T > t_*) = P(T > 2.31)$

Using t-table, P-value is between 0.01 and 0.02.

Exact P-value $= 0.0149$ (using R).

c. Since P-value $< \alpha = 0.05$, we reject H_0.

Example

$H_0 : \mu = 18$
$H_a : \mu \neq 18$
A sample of 48 provided a sample mean $\bar{x} = 17$ and a sample standard deviation $s = 4.5$.
a. Compute the value of the test statistic.
b. Use the t distribution table to compute a range for the p-value.
c. At $\alpha = 0.05$, what is your conclusion?

Solution

a. $t_* = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{17-18}{4.5/\sqrt{48}} = -1.54$

b. Degrees of freedom $= n - 1 = 47$.

P-value $= 2P(T > |t_*|) = 2P(T > |-1.54|) = 2P(T > 1.54)$

Using t-table, P-value is between 0.10 and 0.20.

Exact P-value $= 0.1303$ (using R).

c. Since P-value $> \alpha = 0.05$, we CAN'T reject H_0.

Example: Ancient air

The composition of the earth's atmosphere may have changed over time. To try to discover the nature of the atmosphere long ago, we can examine the gas in bubbles inside ancient amber. Amber is tree resin that has hardened and been trapped in rocks. The gas in bubbles within amber should be a sample of the atmosphere at the time the amber was formed. Measurements on specimens of amber from the late Cretaceous era (75 to 95 million years ago) give these percents of nitrogen:
63.4 65 64.4 63.3 54.8 64.5 60.8 49.1 51.0
Assume (this is not yet agreed on by experts) that these observations are an SRS from the late Cretaceous atmosphere.

Ancient air, continued

Do the data of our Example (see above) give good reason to think that the percent of nitrogen in the air during the Cretaceous era was different from the present 78.1%? Carry out a test of significance at the 5% significance level.

Solution

Is there evidence that the percent of nitrogen in ancient air was different from the present 78.1%?

1. State hypotheses. $H_0 : \mu = 78.1\%$ vs $H_a : \mu \neq 78.1\%$.
2. Test statistic. $t^* = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{59.5888-78.1}{6.2553/\sqrt{9}} = -8.8778$
3. P-value. For $df = 8$, this is beyond anything shown in Table, so P-value < 0.001.
4. Conclusion. Since P-value < 0.001, we reject H_0. We have very strong evidence that Cretaceous air is different from modern air.

```
# Step 1. Entering data;

nitrogen=c(63.4 ,65,64.4,63.3,54.8,
64.5,60.8,49.1,51.0);

# Step 2. Hypothesis test;

t.test(nitrogen,alternative="two.sided", mu=78.1);
```

```
##
##  One Sample t-test
##
## data:  nitrogen
## t = -8.8778, df = 8, p-value = 2.049e-05
## alternative hypothesis: true mean is not equal to 78.1
## 95 percent confidence interval:
##  54.78065 64.39713
## sample estimates:
## mean of x
##  59.58889
```

Example

The Employment and Training Administration reported the U.S. mean unemployment insurance benefit of \$ 238 per week. A researcher in the state of Virginia anticipated that sample data would show evidence that the mean weekly unemployment insurance benefit in Virginia was below the national level.

a. Develop appropriate hypotheses such that rejection of H_0 will support the researcher's contention.

b. For a sample of 100 individuals, the sample mean weekly unemployment insurance benefit was \$231 with a sample standard deviation of \$80. What is the p-value?

c. At $\alpha = 0.05$, what is your conclusion?.

Solution

a. $H_0 : \mu = 238$ vs $H_a : \mu < 238$.
b. $t_* = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{231-238}{80/\sqrt{100}} = -0.88$
Degrees of freedom $= n - 1 = 99$.
Using t table, P-value is between 0.10 and 0.20
c. P-value > 0.05, we CAN'T reject H_0. Cannot conclude mean weekly benefit in Virginia is less than the national mean.

Two-sample problems

- The goal of inference is to compare the responses to two treatments or to compare the characteristics of two populations.
- We have a separate sample from each treatment or each population.

Conditions for inference comparing two means

- We have two SRSs, from two distinct populations. The samples are independent. That is, one sample has no influence on the other. Matching violates independence, for example. We measure the same response variable for both samples.
- Both populations are Normally distributed. The means and standard deviations of the populations are unknown. In practice, it is enough that the distributions have similar shapes and that the data have no strong outliers.

The Two-Sample Procedures

Draw an SRS of size n_1 from a Normal population with unknown mean μ_1, and draw and independent SRS of size n_2 from another Normal population with unknown mean μ_2. A confidence interval for $\mu_1 - \mu_2$ is given by

$$(\bar{x}_1 - \bar{x}_2) \pm t^* \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Here t^* is the critical value for the $t(k)$ density curve with area C between $-t^*$ and t^*. The degrees of freedom k are equal to the smaller of $n_1 - 1$ and $n_2 - 1$.

To test the hypothesis $H_0 : \mu_1 = \mu_2$, calculate the two-sample t statistic

$$t^* = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

and use P-values or critical values for the $t(k)$ distribution.

Degrees of freedom (Option 1)

Option 1. With software, use the statistic t with accurate critical values from the approximating t distribution.

The distribution of the two-sample t statistic is very close to the t distribution with degrees of freedom df given by

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left(\frac{1}{n_1 - 1}\right)\left(\frac{s_1^2}{n_1}\right)^2 + \left(\frac{1}{n_2 - 1}\right)\left(\frac{s_2^2}{n_2}\right)^2}$$

This approximation is accurate when both sample sizes n_1 and n_2 are 5 or larger.

Degrees of freedom (Option2)

Option 2. Without software, use the statistic t with critical values from the t distribution with degrees of freedom equal to the smaller of $n_1 - 1$ and $n_2 - 1$. These procedures are always conservative for any two Normal populations.

Logging in the rain forest

"Conservationists have despaired over destruction of tropical rain forest by logging, clearing, and burning". These words begin a report on a statistical study of the effects of logging in Borneo. Here are data on the number of tree species in 12 unlogged forest plots and 9 similar plots logged 8 years earlier:

Unlogged: 22 18 22 20 15 21 13 13 19 13 19 15

Logged : 17 4 18 14 18 15 15 10 12

Does logging significantly reduce the mean number of species in a plot after 8 years? State the hypotheses and do a t test. Is the result significant at the 5% level?

Solution

Does logging significantly reduce the mean number of species in a plot after 8 years?
1. State hypotheses. $H_0 : \mu_1 = \mu_2$ vs $H_a : \mu_1 > \mu_2$, where μ_1 is the mean number of species in unlogged plots and μ_2 is the mean number of species in plots logged 8 years earlier.
2. Test statistic. $t^* = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = 2.1140$ ($\bar{x}_1 = 17.5$, $\bar{x}_2 = 13.6666$, $s_1 = 3.5290$, $s_2 = 4.5$, $n_1 = 12$ and $n_2 = 9$)
3. P-value. Using Table, we have $df = 8$, and $0.025 <$ P-value < 0.05.
4. Conclusion. Since P-value < 0.05, we reject H_0. There is strong evidence that the mean number of species in unlogged plots is greater than that for logged plots 8 year after logging.

Logging in the rainforest, continued

Use the data from the previous exercise to give a 99% confidence interval for the difference in mean number of species between unlogged and logged plots.

Solution

1. Find $\bar{x}_1 - \bar{x}_2$. From what we did earlier:
$\bar{x}_1 - \bar{x}_2 = 17.5 - 13.6666 = 3.8334$
2. Find SE = Standard Error. We already know that: $s_1 = 3.5290$, $s_2 = 4.5$, $n_1 = 12$ and $n_2 = 9$

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{3.5290^2}{12} + \frac{4.5^2}{9}} = 1.8132$$

3. Find $m = t^* SE$. From Table, we have $df = 8$ and 99% confidence level, then $t^* = 3.355$. Hence, $m = (3.355)(1.8132) = 6.0832$.
4. Find Confidence Interval.
$\bar{x}_1 - \bar{x}_2 \pm t^* SE = 3.8334 \pm 6.0832$ from -2.2498 to 9.9166.

Exercise

In random samples of 25 from each of two Normal populations, we found the following statistics:
$\bar{x}_1 = 524$ and $s_1 = 129$
$\bar{x}_2 = 469$ and $s_2 = 141$
Estimate the difference between the two population means with 95% confidence.

Solution

1. Find $\bar{x}_1 - \bar{x}_2$. In this case: $\bar{x}_1 - \bar{x}_2 = 524 - 469 = 55$

2. Find SE = Standard Error. We already know that: $s_1 = 129$, $s_2 = 141$, $n_1 = 25$ and $n_2 = 25$

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{129^2}{25} + \frac{141^2}{25}} = 38.2215$$

3. Find $m = t^* SE$. From Table, we have $df = 24$ and 95% confidence level, then $t^* = 2.064$. Hence, $m = (2.064)(38.2215) = 78.8892$.

4. Find Confidence Interval.

$\bar{x}_1 - \bar{x}_2 \pm t^* SE = 55 \pm 78.8892$ from -23.8892 to 133.8892.

Exercise

In random samples of 12 from each of two Normal populations, we found the following statistics:
$\bar{x}_1 = 74$ and $s_1 = 18$
$\bar{x}_2 = 71$ and $s_2 = 16$
Test with $\alpha = 0.05$ to determine whether we can infer that the population means differ.

Solution

1. State hypotheses. $H_0 : \mu_1 = \mu_2$ vs $H_a : \mu_1 \neq \mu_2$.
2. Test statistic. $t^* = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = 0.4315$ ($\bar{x}_1 = 74$, $\bar{x}_2 = 71$, $s_1 = 18$, $s_2 = 16$, $n_1 = 12$ and $n_2 = 12$)
3. P-value. Using Table, we have $df = 11$, and P-value > 0.50.
4. Conclusion. Since P-value $> \alpha = 0.05$, we **can't** reject H_0. There is not enough evidence to infer that the population means differ.

Matched pairs t procedures

To compare the responses to the two treatments in a matched pairs design, find the difference between the responses within each pair. Then apply the one-sample t procedures to these differences.
A matched pairs design compares just two treatments. Choose pairs of subjects that are as closely matched as possible. Assign one of the treatments to one of the subjects in a pair by tossing a coin or reading odd and even digits from a table of random digits (or by generating them with a computer). The other subject gets the remaining treatment. Sometimes each "pair" in a matched pairs design consists of just one subject, who gets both treatments one after the other.

Example

A manufacturer wanted to compare the wearing qualities of two different types of automobile tires, A and B. In the comparison, a tire of type A and one of type B were randomly assigned and mounted on the rear wheels of each of five automobiles. The automobiles were then operated for a specified number of miles, and the amount of wear was recorded for each tire. These measurements appear in a table below. Do the data provide sufficient evidence to indicate a difference in mean wear for tire types A and B? Test using $\alpha = 0.05$.

Auto	1	2	3	4	5
Tire A	10.6	9.8	12.3	9.7	8.8
Tire B	10.2	9.4	11.8	9.1	8.3

Solution

You can verify that the mean and standard deviation of the five **difference** measurements are $\bar{d} = 0.48$ and $s_d = 0.0837$.

Step 1. State Hypotheses. $H_0 : \mu_d = 0$ vs $H_a : \mu_d \neq 0$.
Step 2. Find test statistic. $t^* = \frac{\bar{d}-0}{s_d/\sqrt{n}} = \frac{0.48}{0.0837/\sqrt{5}} = 12.8$
Step 3. Compute P-value. Using Table, $P - value < 0.001$
Step 4. Conclusion. Since $P - value < \alpha = 0.05$, we reject H_0. There is ample evidence of a difference in the mean amount of wear for tire types A and B.

Example

Find a 95% confidence interval for $(\mu_A - \mu_B) = \mu_d$ using the data from our previous example.

Solution

A 95% confidence interval for the difference between the mean wear is

$$\bar{d} \pm t^* \frac{s_d}{\sqrt{n}}$$

$$0.48 \pm (2.776)\frac{0.0837}{\sqrt{5}}$$

$$0.48 \pm 0.1039$$

Exercise

In an effort to determine whether a new type of fertilizer is more effective than the type currently in use, researchers took 12 two-acre plots of land scattered throughout the county. Each plot was divided into two equal-size subplots, one of which was treated with the new fertilizer. Wheat was planted, and the crop yields were measured.

Exercise

Plot	1	2	3	4	5	6	7	8	9	10	11	12
Current	56	45	68	72	61	69	57	55	60	72	75	66
New	60	49	66	73	59	67	61	60	58	75	72	68

Exercise

a. Can we conclude at the 5% significance level that the new fertilizer is more effective than the current one?

b. Estimate with 95% confidence the difference in mean crop yields between the two fertilizers.

Solution a)

You can verify that the mean and standard deviation of the twelve **difference** measurements are $\bar{d}$ = new - current = 1 and $s_d = 3.0151$.

Step 1. State Hypotheses. $H_0 : \mu_d = 0$ vs $H_a : \mu_d > 0$.
Step 2. Find test statistic. $t^* = \frac{\bar{d}-0}{s_d/\sqrt{n}} = \frac{1}{3.0151/\sqrt{12}} = 1.1489$
Step 3. Compute P-value. Using Table (df=11), $0.10 < P - value < 0.15$.
Exact P-value = 0.1375, using R.
Step 4. Conclusion. Since $P - value > \alpha = 0.05$, we **can't** reject H_0.
There is not enough evidence to infer that the new fertilizer is better.

R Code

```
# Step 1. Entering data;

current=c(56, 45, 68, 72, 61, 69, 57, 55, 60, 72, 75, 66);

new=c(60, 49, 66, 73, 59, 67, 61, 60, 58, 75, 72, 68);

diff=new-current;

# Step 2. T test;

t.test(diff,alternative="greater");
```

R Code

```
##
##  One Sample t-test
##
## data:  diff
## t = 1.1489, df = 11, p-value = 0.1375
## alternative hypothesis: true mean is greater than 0
## 95 percent confidence interval:
##  -0.5631171        Inf
## sample estimates:
## mean of x
##         1
```

R Code (Another way)

```
# Step 1. Entering data;

current=c(56, 45, 68, 72, 61, 69, 57, 55, 60, 72, 75, 66);

new=c(60, 49, 66, 73, 59, 67, 61, 60, 58, 75, 72, 68);

# Step 2. T test;

t.test(new,current,paired=T, alternative="greater");
```

R Code (Another way)

```
##
##  Paired t-test
##
## data:  new and current
## t = 1.1489, df = 11, p-value = 0.1375
## alternative hypothesis: true difference in means is greate
## 95 percent confidence interval:
##  -0.5631171        Inf
## sample estimates:
## mean of the differences
##                       1
```

Solution b)

A 95% confidence interval for the difference between the mean crop yields between the two fertilizers is

$$\bar{d} \pm t^* \frac{s_d}{\sqrt{n}}$$

$$1 \pm (2.201)\frac{3.0151}{\sqrt{12}}$$

$$1 \pm 1.9157$$

R Code

```
# Finding CI;

t.test(diff,conf.level=0.95);
```

R Code

```
##
##  One Sample t-test
##
## data:  diff
## t = 1.1489, df = 11, p-value = 0.275
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  -0.9157117  2.9157117
## sample estimates:
## mean of x
##         1
```

R Code (Another way)

```
# Finding CI;

t.test(new,current,paired=T,conf.level=0.95);
```

R Code (Another way)

```
##
##  Paired t-test
##
## data:  new and current
## t = 1.1489, df = 11, p-value = 0.275
## alternative hypothesis: true difference in means is not equ
## 95 percent confidence interval:
##  -0.9157117  2.9157117
## sample estimates:
## mean of the differences
##                       1
```

Large-sample confidence interval for comparing two proportions

Draw an SRS of size n_1 from a population having proportion p_1 of successes and draw an independent SRS of size n_2 from another population having proportion p_2 of successes. When n_1 and n_2 are large, an approximate level C confidence interval for $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) \pm z^* SE$$

In this formula the standard error SE of $\hat{p}_1 - \hat{p}_2$ is

$$SE = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

and z^* is the critical value for the standard Normal density curve with area C between $-z^*$ and z^*.

Hypotheses Tests for Two Proportions

To test the hypothesis $H_0 : p_1 = p_2$ first find the pooled proportion $\hat{p}$ of successes in both samples combined. Then compute the z_* statistic,

$z_* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$

In terms of a variable Z having the standard Normal distribution, the approximate P-value for a test of H_0 against

$H_a : p_1 > p_2$: is : $P(Z > z_*)$

$H_a : p_1 < p_2$: is : $P(Z < z_*)$

$H_a : p_1 \neq p_2$: is : $2P(Z > |z_*|)$

Example

A hospital administrator suspects that the delinquency rate in the payment of hospital bills has increased over the past year. Hospital records show that the bills of 48 of 1284 persons admitted in the month of April have been delinquent for more than 90 days. This number compares with 34 of 1002 persons admitted during the same month one year ago. Do these data provide sufficient evidence to indicate an increase in the rate of delinquency in payments exceeding 90 days? Test using $\alpha = 0.10$.

Solution

Let p_1 and p_2 represent the proportions of all potential hospital admissions in April of this year and last year, respectively, that would have allowed their accounts to be delinquent for a period exceeding 90 days, and let $n_1 = 1284$ admissions this year and the $n_2 = 1002$ admissions last year represent independent random samples from these populations.

Solution

Step 1. State Hypotheses. $H_0 : p_1 = p_2$ vs $H_a : p_1 > p_2$
Step 2. Find test statistic. $\hat{p}_1 = \frac{48}{1284} = 0.0374$ and $\hat{p}_2 = \frac{34}{1002} = 0.0339$

$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{48+34}{1284+1002} = 0.0359$

$z_* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = 0.45$

Solution

Step 3. Compute P-value.
$P - value = P(Z > z^*) = P(Z > 0.45) = 1 - P(Z < 0.45) = 0.3264$

Step 4. Conclusion. Since $P - value > \alpha = 0.10$, we **cannot** reject the null hypothesis that $p_1 = p_2$. The data present insufficient evidence to indicate that the proportion of delinquent accounts in April of this year exceeds the corresponding proportion last year.

HW?

These statistics were calculated from two random samples:
$\hat{p}_1 = 0.60 \quad n_1 = 225 \quad \hat{p}_2 = 0.56 \quad n_2 = 225.$
Calculate the P-value of a test to determine whether there is evidence to infer that the population proportions differ.

Example: How to quit smoking

Nicotine patches are often used to help smokers quit. Does giving medicine to fight depression help? A randomized double-blind experiment assigned 244 smokers who wanted to stop to receive nicotine patches and another 245 to receive both a patch and the antidepression drug bupropion. Results: After a year, 40 subjects in the nicotine patch group had abstained from smoking, as had 87 in the patch-plus-drug group. Give a 99% confidence interval for the difference (treatment minus control) in the proportion of smokers who quit.

Solution

$\hat{p}_1 = \frac{40}{244} \approx 0.1639$ and $\hat{p}_2 = \frac{87}{245} \approx 0.3551$.
The standard error is
$SE = \sqrt{\frac{(0.1639)(1-0.1639)}{244} + \frac{(0.3551)(1-0.3551)}{245}} \approx 0.0387$.
The 99% confidence interval is:

$$(0.3551 - 0.1639) \pm 2.576(0.0387)$$

Lower Confidence Limit $= 0.1912 - 0.0996 = 0.0915$
Upper Confidence Limit $= 0.1912 + 0.0996 = 0.2908$

R Code

```
successes=c(87, 40);

totals=c(245, 244);

prop.test(successes,totals, conf.level=0.99,
correct=FALSE);
```

R Code

```
##
##  2-sample test for equality of proportions without continui
##  correction
##
## data:  successes out of totals
## X-squared = 23.2371, df = 1, p-value = 1.432e-06
## alternative hypothesis: two.sided
## 99 percent confidence interval:
##  0.09152484 0.29081039
## sample estimates:
##    prop 1    prop 2
## 0.3551020 0.1639344
```

R Code

```
successes=c(40,87);

totals=c(244,245);

prop.test(successes,totals, conf.level=0.99,
correct=FALSE);
```

R Code

```
##
##  2-sample test for equality of proportions without continui
##  correction
##
## data:  successes out of totals
## X-squared = 23.2371, df = 1, p-value = 1.432e-06
## alternative hypothesis: two.sided
## 99 percent confidence interval:
##  -0.29081039 -0.09152484
## sample estimates:
##    prop 1    prop 2
## 0.1639344 0.3551020
```

How significant is the evidence that the medicine increases the success rate? State hypotheses, calculate a test statistic, use Table A to give its P-value, and state your conclusion. (Use $\alpha = 0.01$)

Solution

Step 1. State Hypotheses. $H_0 : p_1 = p_2$ vs $H_a : p_1 < p_2$
Step 2. Find test statistic. $\hat{p}_1 = \frac{40}{244} = 0.1639$ and $\hat{p}_2 = \frac{87}{245} = 0.3551$

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{40+87}{244+245} = 0.2597$$

$$z_* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = -4.82$$

Solution

Step 3. Compute P-value.
$P - value = P(Z < z^*) = P(Z < -4.82) < 0.0003$

Step 4. Conclusion. Since $P - value < 0.0003 < \alpha = 0.01$, we **reject** the null hypothesis that $p_1 = p_2$. The data provide very strong evidence that bupropion increases success rate.

R Code

```
successes=c(87, 40);

totals=c(245, 244);

prop.test(successes,totals, alternative="greater",
correct=FALSE);
```

R Code

```
##
##  2-sample test for equality of proportions without continui
##  correction
##
## data:  successes out of totals
## X-squared = 23.2371, df = 1, p-value = 7.161e-07
## alternative hypothesis: greater
## 95 percent confidence interval:
##  0.1275385 1.0000000
## sample estimates:
##    prop 1    prop 2
## 0.3551020 0.1639344
```

APPENDIX

Theorem 7.2

Let $Y_1, Y_2, ..., Y_n$ be defined as in Theorem 7.1. Then $Z_i = \frac{Y_i - \mu}{\sigma}$ are independent, standard Normal random variables, $i = 1, 2, ..., n$, and

$$\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma} \right)^2$$

has a χ^2 distribution with n degrees of freedom (df).

Theorem 7.3

Let $Y_1, Y_2, ..., Y_n$ be a random sample from a Normal distribution with mean μ and variance σ^2. Then

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

has a χ^2 distribution with $(n-1)$ df. Also, $\bar{Y}$ and S^2 are independent random variables.

Development

Let $Y_{11}, Y_{12}, ..., Y_{1n_1}$ denote a random sample of size n_1 from a population with a Normal distribution with mean μ_1 and variance σ^2. Also, let $Y_{21}, Y_{22}, ..., Y_{2n_2}$ denote a random sample of size n_2 from a population with a Normal distribution with mean μ_2 and variance σ^2. Then $\bar{Y}_1 - \bar{Y}_2$ has a Normal distribution with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}$. This implies that

$$Z = \frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}} = \frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

has a $N(0,1)$.

Development

The estimator of σ^2 is obtained by pooling the sample data to obtain the *pooled estimator* S_p^2.

$$S_p^2 = \frac{\sum_{i=1}^{n_1}(Y_{1i} - \bar{Y}_1)^2 + \sum_{i=1}^{n_2}(Y_{2i} - \bar{Y}_2)^2}{n_1 + n_2 - 2} = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

where S_i^2 is the sample variance from the ith sample, $i = 1, 2$.

Further,

$$W = \frac{(n_1 + n_2 - 2)S_p^2}{\sigma^2} = \frac{\sum_{i=1}^{n_1}(Y_{1i} - \bar{Y}_1)^2}{\sigma^2} + \frac{\sum_{i=1}^{n_2}(Y_{2i} - \bar{Y}_2)^2}{\sigma^2}$$

is the sum of two independent χ^2-distributed random variables with $(n_1 - 1)$and $(n_2 - 1)$ df, respectively. Thus, W has a χ^2 distribution with $\nu = (n_1 - 1) + (n_2 - 1) = (n_1 + n_2 - 2)$ df. (See Theorems 7.2 and 7.3). We now use the χ^2-distributed variable W and the independent standard normal quantity Z defined above to form a pivotal quantity.

Development

$$T = \frac{Z}{\sqrt{W/\nu}} = \frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

a quantity that by construction has a t distribution with $(n_1 + n_2 - 2)$ df.