STA 256: Statistics and Probability I

Al Nosedal.
University of Toronto.

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My momma always said: "Life was like a box of chocolates. You never know what you're gonna get."

Forrest Gump.

There are situations where one might be interested in more that one random variable. For example, an automobile insurance policy may cover collision and liability. The loss on collision and the loss on liability are random variables.

Definition 5.1

Let Y_1 and Y_2 be **discrete** random variables. The **joint probability function** for Y_1 and Y_2 is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2),$$

where $-\infty < y_1 < \infty, \ -\infty < y_2 < \infty$.

Theorem 5.1

If Y_1 and Y_2 are **discrete** random variables with joint probability function $p(y_1, y_2)$, then
1. $0 \leq p(y_1, y_2) \leq 1$ for all y_1, y_2.
2. $\sum_{y_1, y_2} p(y_1, y_2) = 1$, where the sum is over all values (y_1, y_2) that are assigned nonzero probabilities.

Example 5.1

A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1 and Y_2, the number who select counter 2. Find the joint probability function of Y_1 and Y_2.

Solution

Let the pair (i, j) denote the simple event that the first customer chose counter i and the second customer chose counter j, where i, $j = 1, 2,$ and 3. The sample space consists of $3 \times 3 = 9$ sample points. Under the assumptions given earlier, each sample point is equally likely and has probability $\frac{1}{9}$. The sample space associated with the experiment is

$$S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}.$$

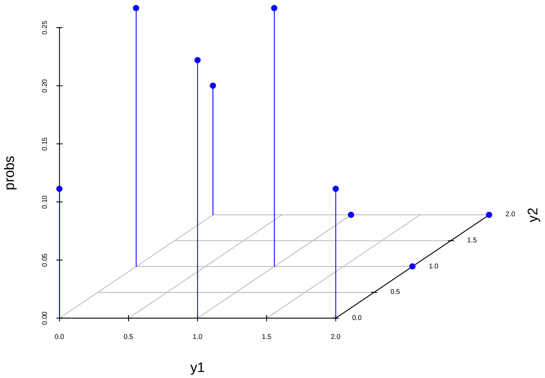
Recall that $Y_1 =$ number of customers who choose counter 1 and $Y_2 =$ number who select counter 2.

Solution (cont.)

Joint probability function for Y_1 and Y_2.

		Y_1	
Y_2	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

Graph of joint probability function



Definition 5.1

Let Y_1 and Y_2 be discrete random variables. The **joint probability function** for Y_1 and Y_2 is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2),$$

where $-\infty < y_1 < \infty$ and $-\infty < y_2 < \infty$.

Definition 5.2

For any random variables Y_1 and Y_2, the **joint distribution function** $F(y_1, y_2)$ is

$$F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2),$$

where $-\infty < y_1 < \infty$ and $-\infty < y_2 < \infty$.

Definition 5.3

Let Y_1 and Y_2 be **continuous** random variables with joint distribution function $F(y_1, y_2)$. If there exists a nonnegative function $f(y_1, y_2)$, such that

$$F(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(t_1, t_2) dt_2 dt_1,$$

for all $-\infty < y_1 < \infty$, $-\infty < y_2 < \infty$, then Y_1 and Y_2 are said to be jointly continuous random variables. The function $f(y_1, y_2)$ is called the **joint probability density function**.

Theorem 5.2

If Y_1 and Y_2 are jointly **continuous** random variables with a joint density function given by $f(y_1, y_2)$, then

1. $f(y_1, y_2) \geq 0$ for all y_1, y_2.
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_2 dy_1 = 1$.

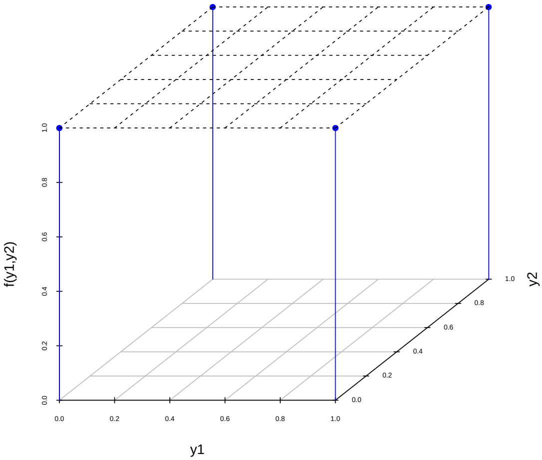
Exercise 5.6

If a radioactive particle is randomly located in a square of unit length, a reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1, & 0 < y_1 < 1,\ 0 < y_2 < 1, \\ 0, & \text{elsewhere} \end{cases}$$

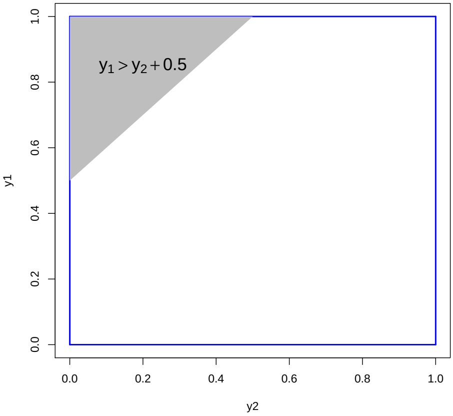
a. What is $P(Y_1 - Y_2 > 0.5)$?
b. What is $P(Y_1 Y_2 < 0.5)$?

Graph of joint probability function



Solution a)

The region $Y_1 > Y_2 + 0.5$ is shown in the figure below.



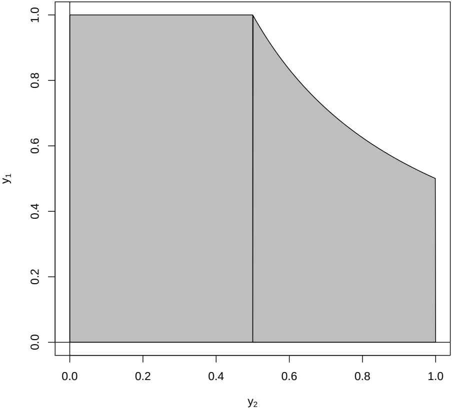
Solution a)

a) $P(Y_1 - Y_2 > 0.5) = P(Y_1 > Y_2 + 0.5) = \int_0^{0.5} \int_{y_2+0.5}^{1} dy_1 dy_2$

$$= \int_0^{0.5} (1 - y_2 - 0.5) dy_2$$
$$= \int_0^{0.5} (0.5 - y_2) dy_2$$
$$= \int_0^{0.5} 0.5 dy_2 - \int_0^{0.5} y_2 dy_2$$
$$= (0.5)^2 - \frac{(0.5)^2}{2} = 0.125$$

Solution b)

The region $Y_1 < 0.5/Y_2$ is shown in the figure below.



Solution b)

b) $P(Y_1 Y_2 < 0.5) = P(Y_1 < \frac{0.5}{Y_2})$

$$= 0.5 + \int_{0.5}^{1} \int_{0}^{0.5/y_2} dy_1 dy_2$$

$$= 0.5 + 0.5 \int_{0.5}^{1} \frac{1}{y_2} dy_2 = 0.5 + 0.5(ln(1) - ln(0.5))$$

$$= 0.5 + 0.5(-ln(0.5)) = 0.5 + 0.3465736 = 0.8465736.$$

Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} ky_1y_2, & 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

a. Find the value of k that makes this a probability density function.

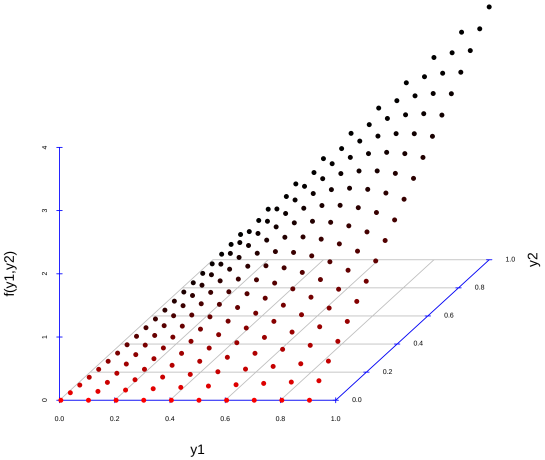
b. Find the joint distribution function for Y_1 and Y_2.

c. Find $P(Y_1 \leq 1/2, Y_2 \leq 3/4)$.

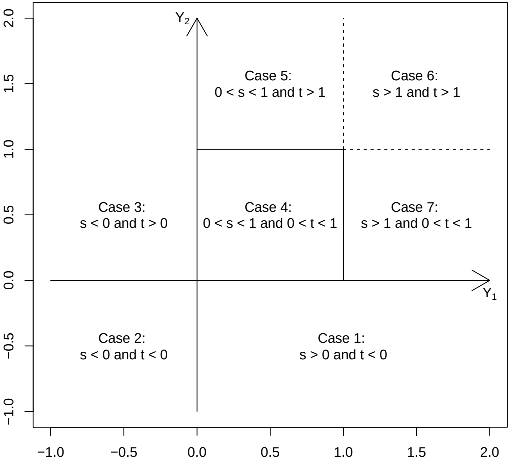
Solution a)

$\int_0^1 \int_0^1 y_1 y_2 dy_1 dy_2 = \int_0^1 y_2 \int_0^1 y_1 dy_1 dy_2$
$\int_0^1 \frac{y_2}{2} dy_2 = \frac{1}{2} \int_0^1 y_2 dy_2 = (1/2)(1/2) = 1/4$
$k = 4$

Graph of joint probability function



Cumulative Distribution Function (cases)



Solution b)

Case 1, Case 2, and Case 3.
$F(s,t) = \int_0^t \int_0^s 0 dy_1 dy_2 = 0$

Solution b)

Case 4. $0 \leq s \leq 1$ and $0 \leq t \leq 1$.

$F(s,t) = \int_0^t \int_0^s 4y_1 y_2 dy_1 dy_2 = 4\int_0^t \int_0^s y_1 y_2 dy_1 dy_2$

$= 4\int_0^t y_2 \int_0^s y_1 dy_1 dy_2$

$= 4\int_0^t y_2 \left(\frac{s^2}{2}\right) dy_2 = \frac{4s^2}{2} \int_0^t y_2 dy_2$

$= 2s^2 \left(\frac{t^2}{2}\right) = s^2 t^2.$

Solution b)

Case 5. $0 \leq s \leq 1$ and $t > 1$.

$$\begin{aligned} F(s,t) &= \int_0^1 \int_0^s 4y_1 y_2 dy_1 dy_2 + \int_1^t \int_0^s 0 dy_1 dy_2 \\ &= 4\int_0^1 \frac{s^2}{2} y_2 dy_2 \\ &= 2s^2\left(\frac{1}{2}\right) \\ &= s^2 \end{aligned}$$

Case 6. $s > 1$ and $t > 1$.

$$\begin{aligned} F(s,t) &= \int_0^1 \int_0^1 4y_1 y_2 dy_1 dy_2 + \int_1^t \int_1^s 0 dy_1 dy_2 \\ &= 1 \end{aligned}$$

Case 7. $s > 1$ and $0 \leq t \leq 1$.

$$\begin{aligned} F(s,t) &= \int_0^t \int_0^1 4y_1y_2 dy_1 dy_2 + \int_0^t \int_1^s 0 dy_1 dy_2 \\ &= 4\int_0^t \frac{1}{2} y_2 dy_2 \\ &= 2\left(\frac{t^2}{2}\right) \\ &= t^2 \end{aligned}$$

Solution c)

$$P(Y_1 \leq 1/2, Y_2 \leq 3/4) = (1/2)^2(3/4)^2 = (1/4)(9/16) = 9/64.$$

Exercise 5.12

Let Y_1 and Y_2 denote the proportions of two different types of components in a sample from a mixture of chemicals used as an insecticide. Suppose that Y_1 and Y_2 have the joint density function given by

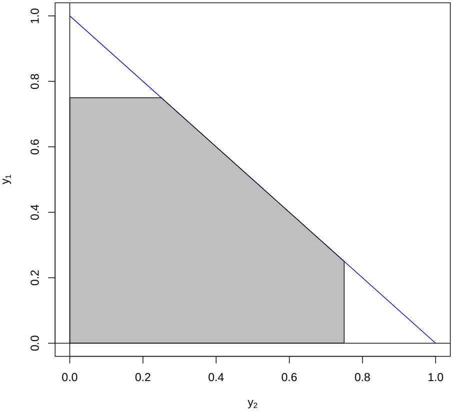
$$f(y_1, y_2) = \begin{cases} 2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

Find

a. $P(Y_1 \leq 3/4, Y_2 \leq 3/4)$.

b. $P(Y_1 \leq 1/2, Y_2 \leq 1/2)$.

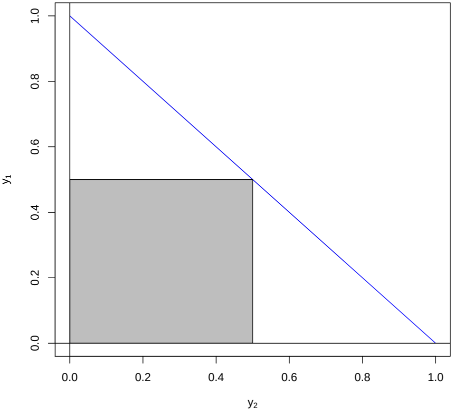
Solution a)



Solution a)

$$P(Y_1 \leq 3/4, Y_2 \leq 3/4) = 1 - (2)(2)(1/2)(1/4)^2$$

Solution b)



Solution b)

$$P(Y_1 \leq 1/2, Y_2 \leq 1/2) = (2)(1/2)^2 = (2)(1/4) = 1/2$$

Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1,\ y_2) = \begin{cases} k(1-y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0 & \text{elsewhere} \end{cases}$$

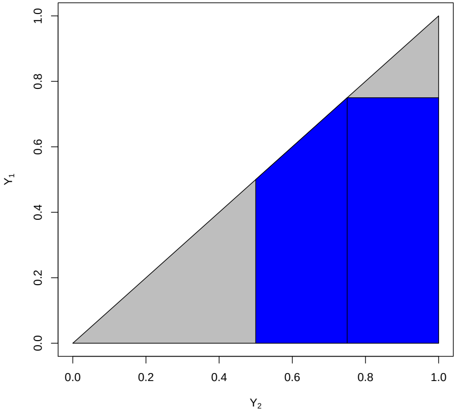
a. Find the value of k that makes this a probability density function.
b. Find $P(Y_1 \leq 3/4,\ Y_2 \geq 1/2)$.

Solution a)

$$\begin{aligned}\int_0^1 \int_0^{y_2} (1-y_2)dy_1 dy_2 &= \int_0^1 (1-y_2) \int_0^{y_2} dy_1 dy_2 \\ &= \int_0^1 (1-y_2)y_2 dy_2 \\ &= \int_0^1 (y_2 - y_2^2) dy_2 \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\end{aligned}$$

Therefore, $k = 6$.

Solution b)



Solution b)

We are interested in the "blue region". We are dividing it in two regions (see previous slide).

$P(Y1 \leq 3/4, Y_2 \geq 1/2) =$

$$\int_{0.5}^{0.75} \int_0^{y_2} 6(1 - y_2) dy_1 dy_2 + \int_{0.75}^{1} \int_0^{0.75} 6(1 - y_2) dy_1 dy_2$$

$$= 6 \left[\int_{0.5}^{0.75} (y_2 - y_2^2) dy_2 + \int_{0.75}^{1} 0.75(1 - y_2) dy_2 \right]$$

$$= 0.484375 = \frac{31}{64}$$

a. Let Y_1 and Y_2 be jointly discrete random variables with probability function $p(y_1, y_2)$. Then the **marginal probability functions** of Y_1 and Y_2, respectively, are given by
$p_1(y_1) = \sum_{\text{all } y_2} p(y_1, y_2)$ and $p_2(y_2) = \sum_{\text{all } y_1} p(y_1, y_2)$.
b. Let Y_1 and Y_2 be jointly continuous random variables with joint density function $f(y_1, y_2)$. Then the **marginal density functions** of Y_1 and Y_2, respectively, are given by
$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_2$ and $f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1$.
The term "marginal" refers to the fact they are the entries in the margins of a table as illustrated by the following example.

Example

The joint probability function of X and Y is given by:

$$\begin{array}{lll} p(1,1) = 0.1 & p(1,2) = 0.2 & p(1,3) = 0.1 \\ p(2,1) = 0.04 & p(2,2) = 0.06 & p(2,3) = 0.1 \\ p(3,1) = 0.05 & p(3,2) = 0.1 & p(3,3) = 0.25 \end{array}$$

Calculate the marginal probability functions of X and Y.

Solution

We can exhibit the values of the joint probability function in the following array:

	j			
i	1	2	3	p(i) = sum along row
1	0.1	0.2	0.1	0.4
2	0.04	0.06	0.1	0.2
3	0.05	0.1	0.25	0.4
p(j) = sum along column	0.19	0.36	0.45	Total = 1

Solution (cont.)

Marginal probability function of X:
$P(X = 1) = 0.4$, $P(X = 2) = 0.2$, $P(X = 3) = 0.4$.

Marginal probability function of Y:
$P(Y = 1) = 0.19$, $P(Y = 2) = 0.36$, $P(Y = 3) = 0.45$.

Example

If $f(x, y) = ce^{-x-2y}$, $x > 0$, $y > 0$ and 0 otherwise, calculate

1. c.
2. The marginal densities of X and Y.
3. $P(1 < X < 2)$.

Solution 1)

Since the joint pdf must integrate to 1,

$$c\int_0^\infty\int_0^\infty e^{-x-2y}dydx = \frac{c}{2}\int_0^\infty e^{-x}dx = \frac{c}{2} = 1.$$

Therefore $c = 2$.

Solution 2)

$$f_X(x) = 2\int_0^{\infty} e^{-x}e^{-2y}\,dy = e^{-x},\ x > 0$$

$$f_Y(y) = 2\int_0^{\infty} e^{-x}e^{-2y}\,dx = 2e^{-2y},\ y > 0.$$

Solution 3)

One way is to get it from the marginal probability density function of X.

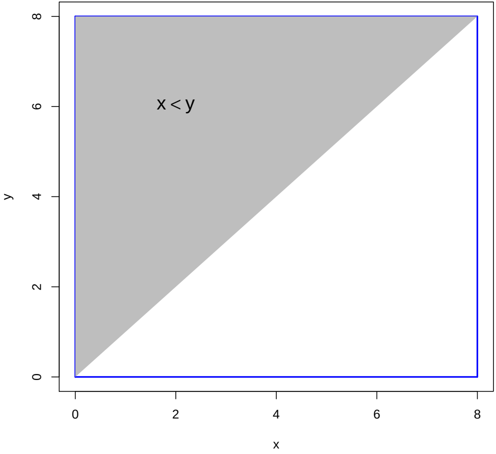
$$P(1 < X < 2) = \int_1^2 f_X(x)dx = \int_1^2 e^{-x}dx = e^{-1} - e^{-2} = 0.2325442.$$

Example

For the random variables in our last example, calculate $P(X < Y)$.

Solution

The region $X < Y$ is shown in the figure below.



Solution

The required probability is the integral of the joint probability density function over this region. From the figure, x goes from 0 to ∞ and for each fixed x, y goes from x to ∞.

$$P(X < Y) = 2\int_0^\infty \int_x^\infty e^{-x-2y}\,dydx = \int_0^\infty e^{-x} \int_x^\infty 2e^{-2y}\,dydx$$

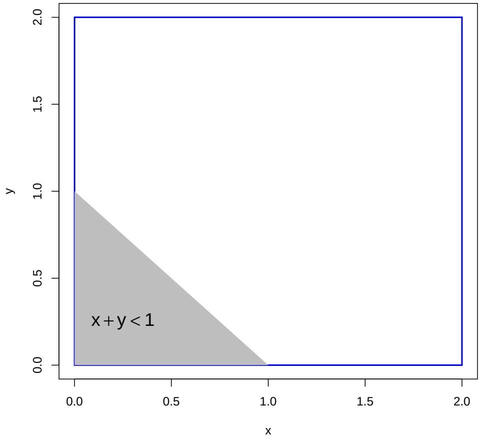
$$P(X < Y) = \int_0^\infty e^{-x}e^{-2x}\,dx = \frac{1}{3}\int_0^\infty 3e^{-3x}\,dx = 1/3.$$

Example

$f(x, y) = 1/4$ if $0 < x < 2$ and $0 < y < 2$.
What is $P(X + Y < 1)$?

Solution

The desired probability is the integral of $f(x, y)$ over the shaded region in the figure below. Let us call this region D.



Solution

$$P(X+Y<1)=\int\int_D \frac{1}{4}dxdy=\frac{1}{4}\text{Area of D}=\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)(1)(1)=\frac{1}{8}.$$

Example

The joint PDF of X and Y is $f(x, y) = cx$, $0 < y < x$ and $0 < x < 2$, 0 elsewhere. Find

1. c.
2. The marginal densities of X and Y.
3. $P(X < 2Y)$.

Solution

1. Since the PDF should integrate to 1,

$$c\int_0^2\int_0^x xdydx = c\int_0^2 x^2dx = c\left(\frac{8}{3}\right) = 1$$

Therefore $c = \frac{3}{8}$.

Solution

2. The marginal density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy.$$

Note that $f(x,y) = 0$ if $y < 0$ or $y > x$ or if $x > 2$. Therefore

$$f_X(x) = \int_0^x \frac{3}{8}xdy = \frac{3}{8}x^2, \ 0 < x < 2$$

and 0 elsewhere.

Solution

Similarly, since the PDF is 0 if $x < y$ or if $x > 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \frac{3}{8}\int_y^2 xdx = \frac{3}{16}(4-y^2),\ 0 < y < 2$$

and 0 elsewhere.

Solution

3. The event $X < 2Y$ corresponds to the region between the lines $y = x$ and $y = \frac{x}{2}$. The probability of it is

$$P(X < 2Y) = \left(\frac{3}{8}\right) \int_0^2 \int_{x/2}^{x} x dy dx = \left(\frac{3}{8}\right) \int_0^2 x(x - x/2) dx$$

$$= \left(\frac{3}{8}\right) \int_0^2 \frac{x^2}{2} dx = \frac{1}{2}.$$

Conditional distributions

Let us first consider the discrete case. Let $p(x, y) = P(X = x, Y = y)$ be the joint PF of the random variables, X and Y. Recall that the conditional probability of the occurrence of event A given that B has occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If A is the event that $X = x$ and B is the event that $Y = y$ then $P(A) = P(X = x) = p_X(x)$, the marginal PF of X, $P(B) = p_Y(y)$, the marginal PF of Y and $P(A \cap B) = p(x, y)$, the joint PF of X and Y.

Conditional distributions (discrete case)

We can then define a conditional probability function for the probability that $X = x$ given $Y = y$ by

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p(x, y)}{p_Y(y)}$$

Similarly

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{p(x, y)}{p_X(x)}$$

Conditional densities (continuous case)

In the continuous case we extend the same concept and define **conditional densities** or conditional PDFs by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

Example

The joint PF of X and Y is given by:

$$\begin{array}{lll} p(1,1) = 0.1 & p(1,2) = 0.2 & p(1,3) = 0.1 \\ p(2,1) = 0.04 & p(2,2) = 0.06 & p(2,3) = 0.1 \\ p(3,1) = 0.05 & p(3,2) = 0.1 & p(3,3) = 0.25 \end{array}$$

Find the conditional PF $p_{X|Y}(x|1)$.

Solution

$p_Y(1) = p(1,1) + p(2,1) + p(3,1) = 0.1 + 0.04 + 0.05 = 0.19.$

$p_{X|Y}(1|1) = \frac{p(1,1)}{p_Y(1)} = \frac{0.1}{0.19} = \frac{10}{19}.$

$p_{X|Y}(2|1) = \frac{p(2,1)}{p_Y(1)} = \frac{0.04}{0.19} = \frac{4}{19}.$

$p_{X|Y}(3|1) = \frac{p(3,1)}{p_Y(1)} = \frac{0.05}{0.19} = \frac{5}{19}.$

Example

If $f(x, y) = 2\exp^{-x-2y}$, $x > 0$, $y > 0$ and 0 otherwise. Find the conditional densities, $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

Solution

Marginals.

$$f_X(x) = 2\int_0^{\infty} e^{-x}e^{-2y}dy = e^{-x}, \ x > 0.$$

$$f_Y(y) = 2\int_0^{\infty} e^{-x}e^{-2y}dx = 2e^{-2y}, \ y > 0.$$

Solution (cont.)

Conditional densities.

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2e^{-x-2y}}{2e^{-2y}} = e^{-x}, \ x > 0.$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2e^{-x-2y}}{e^{-x}} = 2e^{-2y}, \ y > 0.$

Example

The joint PDF of X and Y is $f(x,y) = \frac{3}{8}x$, $0 < y < x$ and $0 < x < 2$, and 0 elsewhere. Calculate the conditional PDFs.

Solution

Marginal densities.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x \frac{3}{8}xdy = \frac{3}{8}x^2, \ 0 < x < 2$$

and 0 elsewhere.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \frac{3}{8}\int_y^2 xdx = \frac{3}{16}(4-y^2), \ 0 < y < 2$$

and 0 elsewhere.

Solution

We have everything we need. We just have to be careful with the domains. For given $Y = y$, $y < x < 2$. For given $X = x$, $0 < y < x < 2$.

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{(3/8)x}{(3/16)(4-y^2)} = \frac{2x}{4-y^2}, y < x < 2,$
and 0 elsewhere.

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{(3/8)x}{(3/8)(x^2)} = \frac{1}{x}, 0 < y < x,$
and 0 elsewhere.

Exercise 5.27

In Exercise 5.9, we determined that

$$f(y_1,\ y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

a. the marginal density functions for Y_1 and Y_2.

b. $P(Y_2 \le 1/2 | Y_1 \le 3/4)$.

c. the conditional density function of Y_1 given $Y_2 = y_2$.

d. the conditional density function of Y_2 given $Y_1 = y_1$.

e. $P(Y_2 \ge 3/4 | Y_1 = 1/2)$.

Solution a)

By definition, $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)dy_2$. In this case,

$$
\begin{aligned}
f_{Y_1}(y_1) &= \int_{y_1}^{1} 6(1 - y_2)dy_2 \quad \text{(recall that } y_1 \leq y_2 \leq 1\text{)} \\
&= 6\left[\int_{y_1}^{1} dy_2 - \int_{y_1}^{1} y_2 dy_2\right] \\
&= 6\left[(1 - y_1) - \frac{1 - y_1^2}{2}\right] \\
&= 3(1 - y_1)^2, \;\; 0 \leq y_1 \leq 1.
\end{aligned}
$$

Solution a)

By definition, $f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2) dy_1$. In this case,

$$
\begin{aligned}
f_{Y_2}(y_2) &= \int_0^{y_2} 6(1 - y_2) dy_1 \quad \text{(recall that } 0 \leq y_1 \leq y_2\text{)} \\
&= 6(1 - y_2) y_2 \\
&= 6(y_2 - y_2^2), \ \ 0 \leq y_2 \leq 1.
\end{aligned}
$$

Solution b)

$$P(Y_2 \leq 1/2 | Y_1 \leq 3/4) = \frac{P(Y_1 \leq 3/4 and Y_2 \leq 1/2)}{P(Y_1 \leq 3/4)}$$
$$= \frac{P(Y_1 \leq 3/4, Y_2 \leq 1/2)}{P(Y_1 \leq 3/4)}$$
$$= \frac{\int_0^{1/2} \int_0^{y_2} 6(1-y_2) dy_1 dy_2}{\int_0^{3/4} 3(1-y_1)^2 dy_1}$$
$$= \frac{1/2}{63/64} = \frac{32}{63}.$$

Solution c) and d)

By definition, $f(y_1|y_2) = \frac{f(y_1,y_2)}{f_{Y_2}(y_2)}$. In this case,

$f(y_1|y_2) = \frac{6(1-y_2)}{6(1-y_2)y_2} = \frac{1}{y_2}$, $0 \leq y_1 \leq y_2 \leq 1$.

Similarly, $f(y_2|y_1) = \frac{f(y_1,y_2)}{f_{Y_1}(y_1)}$. In this case,

$f(y_2|y_1) = \frac{6(1-y_2)}{3(1-y_1)^2} = \frac{2(1-y_2)}{(1-y_1)^2}$, $0 \leq y_1 \leq y_2 \leq 1$.

Solution e)

$$P(Y_2 \geq 3/4 | Y_1 = 1/2) = \int_{3/4}^{1} f(y_2|1/2) dy_2$$
$$= \int_{3/4}^{1} \frac{2(1-y_2)}{(1-1/2)^2} dy_2$$
$$= 8 \int_{3/4}^{1} (1 - y_2) dy_2$$
$$= 8\left(\frac{1}{32}\right) = \frac{8}{32} = \frac{1}{4}.$$

Independent random variables

In one of our examples,

$$f(x, y) = 2\exp^{-x-2y}, f_X(x) = \exp^{-x}, f_Y(y) = 2\exp^{-2y},$$

and so

$$f(x, y) = f_X(x) f_Y(y).$$

If the joint density function is the product of the marginal density functions we say that the random variables are independent.

Theorem 5.4

If Y_1 and Y_2 are discrete random variables with joint probability function $p(y_1, y_2)$ and marginal probability functions $p_1(y_1)$ and $p_2(y_2)$, respectively, then Y_1 and Y_2 are independent if and only if

$$p(y_1, y_2) = p_1(y_1)p_2(y_2)$$

for all pairs of real numbers (y_1, y_2).

Theorem 5.4 (cont.)

If Y_1 and Y_2 are continuous random variables with joint density function $f(y_1, y_2)$ and marginal density functions $f_1(y_1)$ and $f_2(y_2)$, respectively, then Y_1 and Y_2 are independent if and only if

$$f(y_1, y_2) = f_1(y_1)f_2(y_2)$$

for al pairs of real numbers (y_1, y_2).

Exercise 5.43

Let Y_1 and Y_2 have joint density functions $f(y_1, y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$, respectively. Show that Y_1 and Y_2 are independent if and only if $f(y_1|y_2) = f_1(y_1)$ for all values of y_1 and for all y_2 such that $f_2(y_2) > 0$. A completely analogous argument establishes that Y_1 and Y_2 are independent if and only if $f(y_2|y_1) = f_2(y_2)$ for all values of y_2 and for all y_1 such that $f_1(y_1) > 0$.

Proof

Assume that Y_1 and Y_2 are independent. We have to show that
$f(y_1|y_2) = f_1(y_1)$.
$f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)}$ (by definition)
$f(y_1|y_2) = \frac{f_1(y_1)f_2(y_2)}{f_2(y_2)}$ (Y_1 and Y_2 are independent)
$f(y_1|y_2) = f_1(y_1)$.

Proof

Assume $f(y_1|y_2) = f_1(y_1)$. We have to show that Y_1 and Y_2 are independent.
$f(y_1|y_2) = f_1(y_1)$
$\frac{f(y_1,y_2)}{f_2(y_2)} = f_1(y_1)$ (By definition)
$f(y_1, y_2) = f_1(y_1)f_2(y_2)$ (Multiplying both sides by $f_2(y_2)$)
Therefore Y_1 and Y_2 are independent.

Example

If the joint PDF of X and Y is

$$f(x, y) = \frac{1}{8}, \ 0 < x < 4, \ 0 < y < 2, \ \text{and 0 elsewhere.}$$

Determine whether X and Y are independent.

Solution

We have to verify whether or not $f(x,y) = f_X(x)f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^2 \frac{1}{8}dy = \frac{1}{8}(2-0) = \frac{1}{4}, 0 < x < 4,$$

and 0 elsewhere.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^4 \frac{1}{8}dx = \frac{1}{8}(4-0) = \frac{1}{2}, 0 < y < 2,$$

Clearly $f(x,y) = f_X(x)f_Y(y)$. So X and Y are independent.

Example

If the joint PDF of X and Y is given by
$f(x, y) = 2$, $0 < y < x$ and $0 < x < 1$, and 0 elsewhere.
Determine whether or not X and Y are independent.

Solution

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = 2\int_0^x dy = 2x, \;\; 0 < x < 1$, and 0 elsewhere.

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = 2\int_y^1 dx = 2(1-y), \;\; 0 < y < 1$, and 0 elsewhere.

So X and Y are NOT independent.

Exercise 5.53

In Exercise 5.9, we determined that

$$f(y_1,\ y_2) = \begin{cases} 6(1-y_2), & 0 \leq y_1 \leq y_2 \leq 1, \\ 0 & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Are Y_1 and Y_2 independent?

Definition 5.9

Let $g(Y_1, Y_2)$ be a function of the discrete random variables, Y_1 and Y_2, which have probability function $p(y_1, y_2)$. Then the **expected value** of $g(Y_1, Y_2)$ is

$$E[g(Y_1, Y_2)] = \sum_{\text{all } y_1} \sum_{\text{all } y_2} g(y_1, y_2) p(y_1, y_2).$$

If Y_1, Y_2 are continuous random variables with joint density function $f(y_1, y_2)$, then

$$E[g(Y_1, Y_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1, y_2) f(y_1, y_2) dy_1 dy_2.$$

Theorem 5.6

Let c be a constant. Then

$$E(c) = c.$$

Theorem 5.7

Let $g(Y_1, Y_2)$ be a function of the random variables Y_1 and Y_2 and let c be a constant. Then

$$E[cg(Y_1, Y_2)] = cE[g(Y_1, Y_2)].$$

Theorem 5.8

Let Y_1 and Y_2 be random variables and $g_1(Y_1, Y_2)$, $g_2(Y_1, Y_2)$, ... , $g_k(Y_1, Y_2)$ be functions of Y_1 and Y_2. Then

$$E[g_1(Y_1, Y_2) + g_2(Y_1, Y_2) + ... + g_k(Y_1, Y_2)]$$

$$= E[g_1(Y_1, Y_2)] + E[g_2(Y_1, Y_2)] + ... + E[g_k(Y_1, Y_2)].$$

Theorem 5.9

Let Y_1 and Y_2 be **independent** random variables and $g(Y_1)$ and $h(Y_2)$ be functions of only Y_1 and Y_2, respectively. Then

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)],$$

provided that the expectations exist.

Particular case

Let X and Y be **independent** random variables with joint density given by $f(x, y)$. Then

$$E[XY] = E[X]E[Y],$$

provided that the expectations exist.

Proof

By definition 5.9,

$$
\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x)f(y)dxdy \quad \text{(by independence)} \\
&= \int_{-\infty}^{\infty} yf(y)\left[\int_{-\infty}^{\infty} xf(x)dx\right] dy \\
&= \int_{-\infty}^{\infty} yf(y)\left[E(X)\right] dy \quad \text{(by definition of } E(X)\text{)} \\
&= E(X)\int_{-\infty}^{\infty} yf(y)dy \\
&= E(X)E(Y) \quad \text{(by definition of } E(Y)\text{)}
\end{aligned}
$$

Exercise 5.77

In Exercise 5.9, we determined that

$$f(y_1,\ y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0 & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

a. $E(Y_1)$ and $E(Y_2)$.

b. $V(Y_1)$ and $V(Y_2)$.

c. $E(Y_1 - 3Y_2)$.

Solution a)

Using the marginal densities we found in Exercise 5.27, we have that

$E(Y_1) = \int_0^1 3y_1(1 - y_1)^2 dy_1 = \frac{1}{4}$

$E(Y_2) = \int_0^1 6y_2^2(1 - y_2) dy_2 = \frac{1}{2}$

Solution b)

$$E(Y_1^2) = \int_0^1 3y_1^2(1-y_1)^2 dy_1 = \frac{1}{10}$$
$$V(Y_1) = \frac{1}{10} - \left(\frac{1}{4}\right)^2 = \frac{3}{80}.$$

$$E(Y_2^2) = \int_0^1 6y_2^3(1-y_2) dy_2 = \frac{3}{10}$$
$$V(Y_2) = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}.$$

Solution c)

$$E(Y_1 - 3Y_2) = E(Y_1) - 3E(Y_2) = \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}.$$

Whether X and Y are independent or not,

$$E(X+Y) = E(X) + E(Y)$$

Now let us try to calculate $Var(X+Y)$.

$$\begin{aligned} E[(X+Y)^2] &= E[X^2 + 2XY + Y^2] \\ &= E(X^2) + 2E(XY) + E(Y^2)] \end{aligned}$$

$$
\begin{aligned}
Var(X+Y) &= E[(X+Y)^2] - [E(X+Y)]^2 \\
&= E(X^2) + 2E(XY) + E(Y^2) - [E(X)+E(Y)]^2 \\
&= E(X^2) + 2E(XY) + E(Y^2) - [E(X)]^2 - 2E(X)E(Y) - [E(Y)]^2 \\
&= Var(X) + Var(Y) + 2[E(XY) - E(X)E(Y)]
\end{aligned}
$$

Now you can see that the variance of a sum of random variables is **NOT**, in general, the sum of their variances.
If X and Y are independent, however, the last term becomes zero and the variance of the sum is the sum of the variances.
The entity $E(XY) - E(X)E(Y)$ is known as the **covariance** of X and Y.

Definition

If Y_1 and Y_2 are random variables with means μ_1 and μ_2, respectively, the **covariance** of Y_1 and Y_2 is

$$Cov(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)].$$

Definition

It is difficult to employ the covariance as an absolute measure of dependence because its value depends upon the scale of measurement. This problem can be eliminated by standardizing its value and using the **correlation coefficient**, ρ, a quantity related to the covariance and defined as

$$\rho = \frac{Cov(Y_1,\ Y_2)}{\sqrt{V(Y_1)}\sqrt{V(Y_2)}}.$$

Theorem

If Y_1 and Y_2 are random variables with means μ_1 and μ_2, respectively, then

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2).$$

OR

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - \mu_1 \mu_2.$$

Proof

By definition,

$$\begin{aligned}
Cov(Y_1, Y_2) &= E[(Y_1 - \mu_1)(Y_2 - \mu_2)] \\
&= E[Y_1 Y_2 - Y_1 \mu_2 - \mu_1 Y_2 + \mu_1 \mu_2] \\
&= E[Y_1 Y_2] - E[Y_1 \mu_2] - E[\mu_1 Y_2] + E[\mu_1 \mu_2] \\
&= E[Y_1 Y_2] - \mu_2 E[Y_1] - \mu_1 E[Y_2] + \mu_1 \mu_2 \\
&= E[Y_1 Y_2] - \mu_1 \mu_2 - \mu_1 \mu_2 + \mu_1 \mu_2 \\
&= E[Y_1 Y_2] - \mu_1 \mu_2
\end{aligned}$$

Theorem

If Y_1 and Y_2 are independent random variables, then

$$Cov(Y_1, Y_2) = 0.$$

Theorem

Let $Y_1, Y_2, ..., Y_n$ and $X_1, X_2, ..., X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \xi_j$. Define

$$U_1 = \sum_{i=1}^{n} a_i Y_i \quad \text{and} \quad U_2 = \sum_{j=1}^{m} b_j X_j$$

for constants $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_m$. Then the following hold:

a. $E(U_1) = \sum_{i=1}^{n} a_i \mu_i$

b. $Cov(U_1, U_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(Y_i, X_j)$.

Proof

We are going to prove b), when $n = 2$ and $m = 2$.
Let $U_1 = a_1X_1 + a_2X_2$ and $U_2 = b_1Y_1 + b_2Y_2$. Now, recall that

$$cov(U_1, U_2) = E(U_1U_2) - E(U_1)E(U_2).$$

First, we find U_1U_2 and then we compute its expected value.
$U_1U_2 = a_1b_1X_1Y_1 + a_1b_2X_1Y_2 + a_2b_1X_2Y_1 + a_2b_2X_2Y_2$.
$E(U_1U_2) =$
$a_1b_1E(X_1Y_1) + a_1b_2E(X_1Y_2) + a_2b_1E(X_2Y_1) + a_2b_2E(X_2Y_2)$.

Proof (cont.)

Now, we find $E(U_1)E(U_2)$. Clearly, $E(U_1) = a_1\mu_1 + a_2\mu_2$ and $E(U_2) = b_1\xi_1 +_2 \xi_2$. Thus,

$$E(U_1)E(U_2) = a_1 b_1\mu_1\xi_1 + a_1 b_2\mu_1\xi_2 + a_2 b_1\mu_2\xi_1 + a_2 b_2\mu_2\xi_2$$

Proof (cont.)

Finally, $E(U_1 U_2) - E(U_1)E(U_2)$ turns out to be
$a_1 b_1[E(X_1 Y_1) - \mu_1\xi_1] + a_1 b_2[E(X_1 Y_2) - \mu_1\xi_2]$
$+a_2 b_1[E(X_2 Y_1) - \mu_2\xi_1] + a_2 b_2[E(X_2 Y_2) - \mu_2\xi_2]$
which is equivalent to
$a_1 b_1[cov(X_1, Y_1)] + a_1 b_2[cov(X_1, Y_2)]$
$+a_2 b_1[cov(X_2, Y_1)] + a_2 b_2[cov(X_2, Y_2)] = \sum\sum a_i b_j cov(X_i, Y_j)$

Example

Suppose that the random variables X and Y have joint probability density function, $f(x,y)$, given by

$$f(x,y) = \begin{cases} 6(1-y), & 0 \leq x < y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find
a) $E(X)$.
b) $E(Y)$.
c) $E(XY)$.
d) $Cov(X,Y)$.

Solution

a) $E(X) = \int_0^1 \int_x^1 x(6-6y)dydx = \int_0^1 x\left(\int_x^1 6dy - \int_x^1 6ydy\right)$

$= \int_0^1 x\left(6(1-x) - 6[\frac{1-x^2}{2}]\right)$

$= \int_0^1 x(6-6x-3+3x^2)dx$

$= \int_0^1 3x - 6x^2 + 3x^3 dx$

$= \frac{3x^2}{2}|_0^1 \frac{6x^3}{3}|_0^1 + \frac{3x^4}{4}|_0^1$

$= \frac{3}{2} - \frac{6}{3} + \frac{3}{4} = \frac{18-24+9}{12} = \frac{1}{4}.$

Solution

b) $E(Y) = \int_0^1 \int_0^y y(6-6y)dxdy = \int_0^1 (6y-6y^2)(\int_0^y dx)dy$
$= \int_0^1 (6y-6y^2)(y)dy$
$= \int_0^1 (6y^2-6y^3)dy$
$= \frac{6y^3}{3}\Big|_0^1 \frac{6y^4}{4}\Big|_0^1$
$= \frac{6}{3} - \frac{6}{4} = \frac{24-18}{12} = \frac{1}{2}.$

Solution

c) $E(XY) = \int_0^1 \int_0^y (xy)(6 - 6y)dxdy = \int_0^1 \int_0^y (x)(6y - 6y^2)dxdy$
$= \int_0^1 (6y - 6y^2) \left(\int_0^y x dx \right) dy$
$= \int_0^1 (6y - 6y^2) \left(\frac{x^2}{2} |_0^y \right) dy$
$= \int_0^1 (6y - 6y^2) \left(\frac{y^2}{2} \right) dy$
$= \int_0^1 (3y^3 - 3y^4)dy$
$= \frac{3y^4}{4}|_0^1 - \frac{3y^5}{5}|_0^1 = \frac{3}{4} - \frac{3}{5} = \frac{15-12}{20} = \frac{3}{20}.$

Solution

d) $Cov(X,Y) = E(XY) - E(X)E(Y)$
$Cov(X,Y) = \frac{3}{20} - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$
$Cov(X,Y) = \frac{3}{20} - \frac{1}{8} = \frac{6-5}{40} = \frac{1}{40}$.

Exercise 5.89

In Exercise 5.1, we determined that the joint distribution of Y_1, the number of contracts awarded to firm A, and Y_2, the number of contracts awarded to firm B, is given by the entries in the following table.

	y_1		
y_2	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

Find $Cov(Y_1, \ Y_2)$.

Solution

$$E(Y_1) = 0P(Y_1 = 0) + 1P(Y_1 = 1) + 2P(Y_1 = 2)$$
$$E(Y_1) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3}.$$

$$E(Y_2) = 0P(Y_2 = 0) + 1P(Y_2 = 1) + 2P(Y_2 = 2)$$
$$E(Y_2) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3}.$$

Solution

$E(Y_1Y_2) = (0)(0)P(Y_1 = 0,\ Y_2 = 0) + (0)(1)P(Y_1 = 0,\ Y_2 = 1) + ... + (2)(2)P(Y_1 = 2,\ Y_2 = 2)$
$= 0(1/9) + 0(2/9) + 0(1/9) + 0(2/9) + 1(2/9) + 2(0) + 0(1/9) + 2(0) + 4(0)$
$E(Y_1Y_2) = \frac{2}{9}$

Finally, $cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \frac{2}{9} - \frac{4}{9} = -\frac{2}{9}$.

Exercise 5.103

Assume that Y_1, Y_2, and Y_3 are random variables, with
$E(Y_1) = 2, \quad E(Y_2) = -1, \quad E(Y_3) = 4,$
$V(Y_1) = 4, \quad V(Y_2) = 6, \quad V(Y_3) = 8,$
$Cov(Y_1, Y_2) = 1, \quad Cov(Y_1, Y_3) = -1, \quad Cov(Y_2, Y_3) = 0.$
Find $E(3Y_1 + 4Y_2 - 6Y_3)$ and $V(3Y_1 + 4Y_2 - 6Y_3)$.

Solution

$$
\begin{aligned}
E(3Y_1 + 4Y_2 - 6Y_3) &= 3E(Y_1) + 4E(Y_2) - 6E(Y_3) \\
&= 3(2) + 4(-1) - 6(4) \\
&= 6 - 4 - 24 = 6 - 28 = -22.
\end{aligned}
$$

Solution

$$\begin{aligned}
V(3Y_1 + 4Y_2 - 6Y_3) &= V(3Y_1) + V(4Y_2) + V(-6Y_3) \\
&+2Cov(3Y_1, 4Y_2) + 2Cov(3Y_1, -6Y_3) + 2Cov(4Y_2, -6Y_3) \\
&= (3)^2 V(Y_1) + (4)^2 V(Y_2) + (-6)^2 V(Y_3) \\
&+(2)(3)(4) Cov(Y_1, Y_2) + (2)(3)(-6) Cov(Y_1, Y_3) \\
&+(2)(4)(-6) Cov(Y_2, Y_3) \\
&= 9(4) + 16(6) + 36(8) + 24(1) - 36(-1) - 48(0) \\
&= 36 + 96 + 288 + 24 + 36 = 480.
\end{aligned}$$

	$3Y_1$	$4Y_2$	$-6Y_3$
$3Y_1$	$9(Y_1, Y_1)$	$12(Y_1, Y_2)$	$-18(Y_1, Y_3)$
$4Y_2$	$12(Y_1, Y_2)$	$16(Y_2, Y_2)$	$-24(Y_2, Y_3)$
$-6Y_3$	$-18(Y_1, Y_3)$	$-24(Y_2, Y_3)$	$36(Y_3, Y_3)$

	$3Y_1$	$4Y_2$	$-6Y_3$
$3Y_1$	$9cov(Y_1, Y_1)$	$12cov(Y_1, Y_2)$	$-18cov(Y_1, Y_3)$
$4Y_2$	$12cov(Y_1, Y_2)$	$16cov(Y_2, Y_2)$	$-24cov(Y_2, Y_3)$
$-6Y_3$	$-18cov(Y_1, Y_3)$	$-24cov(Y_2, Y_3)$	$36cov(Y_3, Y_3)$

	$3Y_1$	$4Y_2$	$-6Y_3$
$3Y_1$	$9V(Y_1)$	$12Cov(Y_1, Y_2)$	$-18Cov(Y_1, Y_3)$
$4Y_2$	$12Cov(Y_1, Y_2)$	$16V(Y_2)$	$-24Cov(Y_2, Y_3)$
$-6Y_3$	$-18Cov(Y_1, Y_3)$	$-24Cov(Y_2, Y_3)$	$36V(Y_3)$

$$\begin{aligned} V(3Y_1 + 4Y_2 - 6Y_3) &= 9V(Y_1) + 16V(Y_2) + 36V(Y_3) \\ &+ 24Cov(Y_1, Y_2) - 36Cov(Y_1, Y_3) \\ &- 48Cov(Y_2, Y_3) \end{aligned}$$

$$
\begin{aligned}
V(3Y_1 + 4Y_2 - 6Y_3) &= 9V(Y_1) + 16V(Y_2) + 36V(Y_3) \\
&+ 24Cov(Y_1, Y_2) - 36Cov(Y_1, Y_3) \\
&- 48Cov(Y_2, Y_3)
\end{aligned}
$$

$$
\begin{aligned}
V(3Y_1 + 4Y_2 - 6Y_3) &= 9(4) + 16(6) + 36(8) \\
&+ 24(1) - 36(-1) \\
&- 48(0)
\end{aligned}
$$

$$
\begin{aligned}
V(3Y_1 + 4Y_2 - 6Y_3) &= 36 + 96 + 288 \\
&+ 24 + 36 - 0 \\
&= 480
\end{aligned}
$$

We would like to invest \$10,000 into shares of companies XX and YY. Shares of XX cost \$20 per share. The market analysis shows that their expected return is \$1 per share with a standard deviation of \$0.5. Shares of YY cost \$50 per share, with an expected return of \$2.50 and a standard deviation of \$1 per share, and returns from the two companies are independent. In order to maximize the expected return and minimize the risk (standard deviation or variance), is it better to invest (A) all \$10, 000 into XX, (B) all \$10,000 into YY, or (C) \$5,000 in each company?

Solution (A)

Let X be the actual (random) return from each share of XX, and Y be the actual return from each share of YY. Compute the expectation and variance of the return for each of the proposed portfolios (A, B, and C)

At \$20 a piece, we can use \$10,000 to buy 500 shares of XX, thus $A = 500X$.

$$E(A) = 500E(X) = (500)(1) = 500;$$

$$V(A) = 500^2 V(X) = 500^2(0.5)^2 = 62,500.$$

Solution (B)

Investing all \$10,000 into YY, we buy $10,000/50 = 200$ shares of it, so that $B = 200Y$,

$$E(B) = 200E(Y) = (200)(2.50) = 500;$$

$$V(B) = 200^2 V(Y) = 200^2(1)^2 = 40,000.$$

Investing \$5,000 into each company makes a portfolio consisting of 250 shares of XX and 100 shares of YY, so that $C = 250X + 100Y$. Since independence yields uncorrelation,

$$E(C) = 250E(X) + 100E(Y) = 250(1) + 100(2.5) = 250 + 250 = 500;$$

$$V(C) = 250^2 V(X) + 100^2 V(Y) = 250^2 (0.5)^2 + 100^2 (1)^2 = 25,625.$$

Solution

The expected return is the same for each of the proposed three portfolios because each share of each company is expected to return $\frac{500}{10{,}000} = \frac{1}{20}$, which is 5%. In terms of the expected return, all three portfolios are equivalent. Portfolio C, where investment is split between two companies, has the lowest variance, therefore, it is the least risky. This supports one of the basic principles in finance: **to minimize the risk, diversify the portfolio**.

Definition 5.13

If Y_1 and Y_2 are any two random variables, the **conditional expectation** of $g(Y_1)$, given that $Y_2 = y_2$, is defined to be

$$E[g(Y_1)|Y_2 = y_2] = \int_{-\infty}^{\infty} g(y_1)f(y_1|y_2)dy_1$$

if Y_1 and Y_2 are jointly continuous and

$$E[g(Y_1)|Y_2 = y_2] = \sum_{\text{all } y_1} g(y_1)p(y_1|y_2)$$

if Y_1 and Y_2 are jointly discrete.

Theorem 5.14

Let Y_1 and Y_2 denote random variables. Then

$$E(Y_1) = E[E(Y_1|Y_2)],$$

where on the right-hand side the inside expectation is with respect to the conditional distribution of Y_1 given Y_2 and the outside expectation is with respect to the distribution of Y_2.

Theorem 5.15

Let Y_1 and Y_2 denote random variables. Then

$$V(Y_1) = E[V(Y_1|Y_2)] + V[E(Y_1|Y_2)].$$

Assume that Y denotes the number of bacteria per cubic centimeter in a particular liquid and that Y has a Poisson distribution with parameter x. Further assume that x varies from location to location and has an exponential distribution with parameter $\beta = 1$.

a) Find $f(x, y)$, the joint probability function of X and Y.
b) Find $f_Y(y)$, the marginal probability function of Y.
c) Find $E(Y)$.
d) Find $f(X|Y = y)$.
e) Find $E(X|Y = 0)$.

Solution

a) $f(x,y) = f(y|x)f_X(x)$
$f(x,y) = \left(\frac{x^y e^{-x}}{y!}\right)(e^{-x})$
$f(x,y) = \frac{x^y e^{-2x}}{y!}$
where $x > 0$ and $y = 0, 1, 2, 3, \ldots$.

Solution

b) $f_Y(y) = \int_0^\infty \frac{x^y e^{-2x}}{y!} dx$
$= \frac{1}{y!} \int_0^\infty x^y e^{-2x} dx$

(We note that $x^y e^{-2x}$ is "almost" a Gamma pdf with $\alpha = y + 1$ and $\beta = 1/2$).

$= \frac{\Gamma(y+1)(1/2)^{y+1}}{y!} \int_0^\infty \frac{1}{\Gamma(y+1)(1/2)^{y+1}} x^y e^{-2x} dx$
$= \frac{\Gamma(y+1)(1/2)^{y+1}}{y!}$

(Recalling that $\Gamma(N) = (N-1)!$ provided that N is a positive integer).

$f_Y(y) = \left(\frac{1}{2}\right)^{y+1}$
where $y = 0, 1, 2, 3, ...$

Solution (using theorem)

c) $E(Y) = E\left(E(Y|X)\right) = E(X) = 1$

Solution (by definition)

c) By definition,

$$
\begin{aligned}
E[Y] &= \sum_{y=0}^{\infty} y \left(\frac{1}{2}\right)^{y+1} \quad \text{(first term is zero)} \\
&= \sum_{x=1}^{\infty} y \left(\frac{1}{2}\right)^{y+1} \\
&= \left(\frac{1}{2}\right) \sum_{y=1}^{\infty} y \left(\frac{1}{2}\right)^{y} \\
&= \left(\frac{1}{2}\right) \sum_{y=1}^{\infty} y \left(\frac{1}{2}\right)^{y-1+1} \quad \text{multiplying by "one"} \\
&= \left(\frac{1}{2}\right) \sum_{y=1}^{\infty} y \left(\frac{1}{2}\right)^{y-1} \left(\frac{1}{2}\right)
\end{aligned}
$$

Solution (by definition)

Note that $\sum_{y=1}^{\infty} y \left(\frac{1}{2}\right)^{y-1} \left(\frac{1}{2}\right)$ is the "formula" you would use to find the expected value of a Geometric random variable with parameter $p = \frac{1}{2}$. Therefore,

$$\begin{aligned} E[Y] &= \left(\frac{1}{2}\right)\left(\frac{1}{1/2}\right) \quad \text{(we know this from table)} \\ &= \left(\frac{1}{2}\right)(2) = 1. \end{aligned}$$

Solution

d) $f(x|y) = \frac{f(x,y)}{f_Y(y)}$

$f(x|y) = \frac{2^{y+1} x^y e^{-2x}}{y!}$

where $x > 0$.

Solution

e) Note that $f(x|y=0) = \frac{2^{0+1}x^0 e^{-2x}}{0!} = 2e^{-2x}$

$$
\begin{aligned}
E(X|Y=0) &= \int_0^\infty x[2e^{-2x}]dx \\
&= 2\int_0^\infty xe^{-2x}dx \\
&= 2\Gamma(2)(1/2)^2 \int_0^\infty \frac{1}{\Gamma(2)(1/2)^2} x^{2-1} e^{-x/(1/2)} dx \\
&= 2\Gamma(2)(1/2)^2 = 2\left(\frac{1}{4}\right) = \frac{1}{2}
\end{aligned}
$$

(Note. Try doing this by parts, too).

Exercise 5.141

Let X have an exponential distribution with mean λ and the conditional density of Y given $X = x$ be

$$f(y|\, x) = \begin{cases} \frac{1}{x}, & 0 \leq y \leq x \\ 0 & \text{elsewhere.} \end{cases}$$

Find $E(Y)$ and $V(Y)$, the unconditional mean and variance of Y.

Solution (using joint density function)

Recall that $f(y|\ x) = \frac{f(x,y)}{f(x)}$. Therefore, $f(x,y) = f(y|\ x)f(x)$.

$$f(x,y) = \begin{cases} \frac{1}{\lambda}x^{-1}e^{-x/\lambda}, & 0 \leq y \leq x \text{ and } x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Solution (using joint density function)

By definition (5.9),

$$
\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dydx \\
&= \int_{0}^{\infty}\int_{0}^{x} y\left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right]dydx \\
&= \int_{0}^{\infty}\left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right]\left[\int_{0}^{x} ydy\right]dx \\
&= \int_{0}^{\infty}\left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right]\left[\frac{y^2}{2}\right]_{0}^{x}dx \\
&= \int_{0}^{\infty}\left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right]\left[\frac{x^2}{2}\right]dx
\end{aligned}
$$

Solution (using joint density function)

$$
\begin{aligned}
E(Y) &= \frac{1}{2\lambda}\int_0^\infty x e^{-x/\lambda} dx \\
&= \frac{1}{2\lambda}\int_0^\infty x^{2-1} e^{-x/\lambda} dx \text{ (multiplying by "one")} \\
&= \frac{\lambda^2\Gamma(2)}{2\lambda}\int_0^\infty \frac{1}{\lambda^2\Gamma(2)} x^{2-1} e^{-x/\lambda} dx \\
&= \frac{\lambda}{2}
\end{aligned}
$$

Solution (using joint density function)

By definition (5.9),

$$
\begin{aligned}
E(Y^2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)dydx \\
&= \int_{0}^{\infty}\int_{0}^{x} y^2 \left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right] dydx \\
&= \int_{0}^{\infty} \left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right]\left[\int_{0}^{x} y^2 dy\right] dx \\
&= \int_{0}^{\infty} \left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right]\left[\frac{y^3}{3}\right]_0^x dx \\
&= \int_{0}^{\infty} \left[\frac{1}{\lambda}x^{-1}e^{-x/\lambda}\right]\left[\frac{x^3}{3}\right] dx
\end{aligned}
$$

Solution (using joint density function)

$$
\begin{aligned}
E(Y^2) &= \frac{1}{3\lambda}\int_0^\infty x^2 e^{-x/\lambda} dx \\
&= \frac{1}{3\lambda}\int_0^\infty x^{3-1} e^{-x/\lambda} dx \text{ (multiplying by "one")} \\
&= \frac{\lambda^3\Gamma(3)}{3\lambda}\int_0^\infty \frac{1}{\lambda^3\Gamma(3)} x^{3-1} e^{-x/\lambda} dx \\
&= \frac{2}{3}\lambda^2
\end{aligned}
$$

Solution (using joint density function)

Using the fact that $V(Y) = E(Y^2) - [E(Y)]^2$.

$$\begin{aligned} V(Y) &= \frac{2}{3}\lambda^2 - \left[\frac{\lambda}{2}\right]^2 \\ &= \frac{2}{3}\lambda^2 - \frac{1}{4}\lambda^2 \\ &= \frac{5}{12}\lambda^2 \end{aligned}$$

Solution (using theorems)

Recalling that $E(Y) = E[E(Y|X)]$ and using the fact that Y given $X = x$ has a uniform probability distribution on the interval $[0, x]$, we have

$$\begin{aligned} E(Y) &= E[E(Y|X)] \\ &= E\left[\frac{X}{2}\right] \quad \text{(from our table, for instance)} \\ &= \frac{1}{2}E[X] \\ &= \frac{\lambda}{2} \quad \text{(from our table, again)} \end{aligned}$$

Solution (using theorems)

Recalling that $V(Y) = E[V(Y|X)] + V[E(Y|X)]$ and using the fact that Y given $X = x$ has a uniform probability distribution on the interval $[0, \ x]$, we have

$$\begin{aligned} V(Y) &= E[V(Y|X)] + V[E(Y|X)] \\ &= E\left[\frac{X^2}{12}\right] + V\left[\frac{X}{2}\right] \quad \text{(from our table)} \\ &= \frac{1}{12}E[X^2] + \frac{1}{4}V[X] \end{aligned}$$

Solution (using theorems)

$V(X) = E[X^2] - [E(X)]^2$, Right? Then,
$E[X^2] = V(X) + [E(X)]^2$. Using this fact, we have

$$
\begin{aligned}
V(Y) &= \frac{1}{12}E[X^2] + \frac{1}{4}V[X] \text{ (from our table)} \\
&= \frac{1}{12}[\lambda^2 + \lambda^2] + \frac{1}{4}\lambda^2 \\
&= \frac{5}{12}\lambda^2.
\end{aligned}
$$