STA 256: Statistics and Probability I

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My momma always said: "Life was like a box of chocolates. You never know what you're gonna get."

Forrest Gump.

Experiment, outcome, sample space, and sample point

When you toss a coin. It comes up heads or tails. Those are the only possibilities we allow. Tossing the coin is called an **experiment**. The results, namely H (heads) and T (tails) are called **outcomes**. There are only two outcomes here and none other. This set of outcomes namely, $\{H, T\}$ is called a **sample space**. Each of the outcomes H and T is called a **sample point**.

Sample space

Now suppose that you toss a coin twice in succession. Then there are four possible outcomes, HH, HT, TH, TT. These are the sample points in the sample space {HH, HT, TH, TT}.
We will denote the sample space by **S**.

Event

An **event** is a set of sample points. In the example of tossing a coin twice in succession, the event, "the first toss results in heads", is the set $\{HH, HT\}$.
Let $\mathbf{A}$ and $\mathbf{B}$ be events. By $\mathbf{A} \subset \mathbf{B}$ (read, " $\mathbf{A}$ is a subset of $\mathbf{B}$ ") we mean that every point that is in $\mathbf{A}$ is also in $\mathbf{B}$. If $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{B} \subset \mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ have to consist of the same points. In that case we write $\mathbf{A} = \mathbf{B}$.

Definitions

- An **experiment** is the process by which an observation is made.
- A **simple event** is an event that cannot be decomposed. Each simple event corresponds to one and only one **sample point**. The letter **E** with a subscript will be used to denote a simple event or the corresponding sample point.
- The **sample space** associated with an experiment is the set consisting of all possible sample points. A sample space will be denoted by **S**.

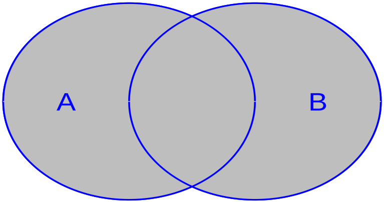
Definitions

- A **discrete sample space** is one that contains either a finite or a countable number of distinct sample points.
- An **event** in a discrete random sample space **S** is a collection of sample points - that is, any subset of **S**.

Union of events

If **A** and **B** are events, then, "**A** or **B**", is also an event denoted by $\mathbf{A} \cup \mathbf{B}$. For example, let **A** be: "the first toss results in heads", and **B** be: "the second toss results in tails", is the set {HH, HT, TT}. Thus $\mathbf{A} \cup \mathbf{B}$ consists of all points that are either in **A** or in **B**.

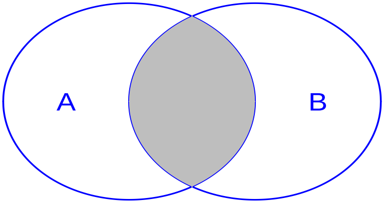
$A \cup B$



Intersection of events

The event "**A** and **B**", is denoted by $\mathbf{A} \cap \mathbf{B}$. For example, if **A** and **B** are the events defined above, then $\mathbf{A} \cap \mathbf{B}$ is the event: "the first toss results in heads and the second toss results in tails", which is the set $\{HT\}$. Thus $\mathbf{A} \cap \mathbf{B}$ consists of all points that belong to both **A** and **B**.

$A \cap B$



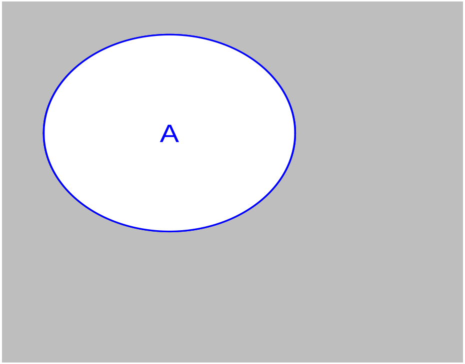
Empty set

The empty set, denoted by $\emptyset$, contains **no** points. It is the impossible event. For instance, if $\mathbf{A}$ is the event that the first toss results in heads and $\mathbf{C}$ is the event that the first toss results in tails, then $\mathbf{A} \cap \mathbf{C}$ is the event that "the first toss results in heads and the first toss results in tails". It is the empty set.
If $\mathbf{A} \cap \mathbf{C} = \emptyset$ we say that the events $\mathbf{A}$ and $\mathbf{C}$ are **mutually exclusive**. That means there are no sample points that are common to $\mathbf{A}$ and $\mathbf{C}$. The events $\mathbf{A}$ and $\mathbf{C}$ in the last paragraph are mutually exclusive.

Complement

The event that "**A** does not occur" is called the **complement** of **A** and is denoted by $\mathbf{A^c}$. It consists of all points that are not in **A**. If **A** is the event that the first toss results in heads, then $\mathbf{A^c}$ is the event that the first toss does not result in heads. It is the set: $\{TH, TT\}$.
Note that $\mathbf{A} \cup \mathbf{A^c} = \mathbf{S}$ and $\mathbf{A} \cap \mathbf{A^c} = \emptyset$.

A^c



Example

A survey is made of a population to find out how many of them own a home, how many own a car and how many are married. Let **H**, **C** and **M** stand respectively for the events, owning a home, owning a car and being married. What do the following symbols represent?

1. $\mathbf{H} \cap \mathbf{M^c}$.
2. $(\mathbf{H} \cup \mathbf{M})^{\mathbf{c}}$.
3. $\mathbf{H^c} \cap \mathbf{M^c}$.
4. $(\mathbf{H} \cup \mathbf{M}) \cap \mathbf{C}$.

Solution

1. $\mathbf{H} \cap \mathbf{M^c}$ corresponds to owning home and not married.
2. The event $(\mathbf{H} \cup \mathbf{M})^{\mathbf{c}}$ it is the event of neither owning a home nor being married.
3. This is the event of not owning home and not married. Note that this is the same as the event (2). That is
$(\mathbf{H} \cup \mathbf{M})^{\mathbf{c}} = \mathbf{H}^c \cap \mathbf{M}^c$.
4. $(\mathbf{H} \cup \mathbf{M}) \cap \mathbf{C}$ corresponds to owning a home or married and owning car.

DeMorgan's Laws

- $(A \cap B)^c = A^c \cup B^c$
- $(A \cup B)^c = A^c \cap B^c$

(Please, do Exercise 2.3.)

If you draw an M & M candy at random from a bag of the candies, the candy you draw will have one of the seven colors. The probability of drawing each color depends on the proportion of each color among all candies made. Here is the distribution for milk chocolate M & M's:

Color	Purple	Yellow	Red
Probability	0.2	0.2	0.2

Color	Orange	Brown	Green	Blue
Probability	0.1	0.1	0.1	?

a) What must be the probability of drawing a blue candy?
b) What is the probability that you do not draw a brown candy?
c) What is the probability that the candy you draw is either yellow, orange, or red?

Solution

a. Probability of Blue $=$ P(Blue) $= 1 - 0.9 = 0.1$
b. P(Not Brown) $=$ 1 - P(Brown) $= 1 - 0.1 = 0.9$
c. P(Yellow or Orange or Red) $= 0.2 + 0.1 + 0.2 = 0.5$

Definition 2.6

Suppose $\mathbf{S}$ is a sample space associated with an experiment. To every event $\mathbf{A}$ in $\mathbf{S}$, we assign a number, $P(\mathbf{A})$, called the probability of $\mathbf{A}$, so that the following axioms hold:

Axiom 1: $0 \leq P(\mathbf{A}) \leq 1$

Axiom 2: $P(\mathbf{S}) = 1$

Axiom 3: If $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ form a sequence of pairwise mutually exclusive events in $\mathbf{S}$ (that is, $\mathbf{A}_i \cap \mathbf{A}_j = \emptyset$ if $i \neq j$), then

$$P(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3 \cup \ldots) = \sum_{i=1}^{\infty} P(\mathbf{A}_i).$$

Exercise 2.9

Every person's blood type is A, B, AB, or O. In addition, each individual either has the Rhesus (Rh) factor $(+)$ or does not $(-)$. A medical technician records a person's blood type and Rh factor. List the sample space for this experiment.

Solution

$$\mathbf{S} = \{A+, B+, AB+, O+, A-, B-, AB-, O-\}$$

A sample space consists of five simple events, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{E}_4$, and $\mathbf{E}_5$.
a. If $P(\mathbf{E}_1) = P(\mathbf{E}_2) = 0.15$, $P(\mathbf{E}_3) = 0.4$, and $P(\mathbf{E}_4) = 2P(\mathbf{E}_5)$, find the probabilities of $\mathbf{E}_4$ and $\mathbf{E}_5$.
b. If $P(\mathbf{E}_1) = 3P(\mathbf{E}_2) = 0.3$, find the probabilities of the remaining simple events if you know that the remaining simple events are equally probable.

Solution

a. $P(\mathbf{S}) = P(\mathbf{E}_1) + P(\mathbf{E}_2) + P(\mathbf{E}_3) + P(\mathbf{E}_4) + P(\mathbf{E}_5) = 1$
$0.15 + 0.15 + 0.40 + 3P(\mathbf{E}_5) = 1$
So, $P(\mathbf{E}_5) = 0.10$ and $P(\mathbf{E}_4) = 0.20$.

b. $P(\mathbf{S}) = P(\mathbf{E}_1) + P(\mathbf{E}_2) + P(\mathbf{E}_3) + P(\mathbf{E}_4) + P(\mathbf{E}_5) = 1$
If $P(\mathbf{E}_1) = 3P(\mathbf{E}_2) = 0.3$, then $P(\mathbf{E}_2) = 0.10$. Which implies that $P(\mathbf{E}_3) + P(\mathbf{E}_4) + P(\mathbf{E}_5) = 0.6$. Thus, they are all equal to 0.2.

Exercise 2.29

Two additional jurors are needed to complete a jury for a criminal trial. There are six prospective jurors, two women and four men. Two jurors are randomly selected from the six available.
a. Define the experiment and describe one sample point. Assume that you need describe only the two jurors chosen and not the order in which they were selected.
b. List the sample space associated with this experiment.
c. What is the probability that both of the jurors selected are women?

Solution

a. The experiment consists of randomly selecting two jurors from a group of two women and four men.

b. Denoting the women as $w_1, \ w_2$ and the men as $m_1, \ m_2, \ m_3, \ m_4$, the sample space is

$(w_1, m_1), (w_1, m_2), (w_1, m_3), (w_1, m_4)$
$(w_2, m_1), (w_2, m_2), (w_2, m_3), (w_2, m_4)$
$(m_1, m_2), (m_1, m_3), (m_1, m_4)$
$(m_2, m_3), (m_2, m_4)$
(m_3, m_4)
(w_1, w_2)

c. $P(w_1, w_2) = 1/15$

The Sample-Point Method

The following steps are used to find the probability of an event:

- Define the experiment and clearly determine how to describe one simple event.
- List the sample events associated with the experiment and test each to make certain that it cannot be decomposed. This defines the sample space **S**.
- Assign reasonable probabilities to the sample points in **S**, making certain that $P(\mathbf{E}_i) \geq 0$ and $\sum P(\mathbf{E}_i) = 1$.
- Define the event of interest, **A**, as a specific collection of sample points.
- Find $P(\mathbf{A})$ by summing the probabilities of the sample points in **A**.

Theorem 2.1

With m elements $a_1,\ a_2, ...,\ a_m$ and n elements $b_1,\ b_2, ...,\ b_n$, it is possible to form $mn = m \times n$ pairs containing one element from each group.

Proof.

We can use a table or a tree diagram to verify this. Please, see page 41.

Problem

How many different committees consisting of a president and a secretary can be selected from a group of five individuals?

Solution

(A,B),(A,C),(A,D),(A,E)
(B,A),(B,C),(B,D),(B,E)
(C,A),(C,B),(C,D),(C,E)
(D,A),(D,B),(D,C),(D,E)
(E,A),(E,B),(E,C),(E,D)
20 committees.

Definition

An **ordered** arrangement of r distinct objects is called a **permutation**. The number of ways of ordering n distinct objects taken r at a time will be designated by the symbol P_r^n.

Theorem 2.2

$$P_r^n = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}.$$

Example

Suppose that an assembly operation in a manufacturing plant involves four steps, which can be performed in any sequence. If the manufacturer wishes to compare the assembly time for each of these sequences, how many different sequences will be involved in the experiment?

Solution

The total number of sequences equals the number of ways of arranging $n = 4$ steps taken $r = 4$ at a time, or

$$P_4^4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4! = (4)(3)(2)(1) = 24.$$

Problem

How many teams consisting of two individuals can be selected from a group of five individuals?

Solution

{A,B},{A,C},{A,D},{A,E}, {B,C},{B,D},{B,E}, {C,D},{C,E}, {D,E}.
10 teams.

Definition

The number of **combinations** of n objects taken r at a time is the number of subsets, each of size r, that can be formed from the n objects. This number will be denoted by C_r^n.

Theorem 2.4

The number of **unordered** subsets of size r chosen (without replacement) from n available objects is

$$C_r^n = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Problem

The Powerball lottery is played twice each week in 28 states, the Virgin Islands, and the District of Columbia. To play Powerball a participant must purchase a ticket and then select five numbers from the digits 1 through 55 and a Powerball number from the digits 1 through 42. To determine the winning numbers for each game, lottery officials draw five white balls out of a drum with 55 white balls, and one red ball out of a drum with 42 red balls. To win the jackpot, a participant's numbers must match the numbers on the five white balls in any order and the number on the red Powerball.

Problem (cont.)

Eight coworkers at the ConAgra Foods plant in Lincoln, Nebraska, claimed the record \$ 365 million jackpot on February 18, 2006, by matching the numbers 15-17-43-44-49 and the Powerball number 29. A variety of other cash prizes are awarded each time the game is played. For instance, a prize of \$ 200,000 is paid if the participant's five numbers match the numbers on the five white balls.

a. Compute the number of ways the first five numbers can be selected.

b. What is the probability of winning a prize of \$200,000 by matching the numbers on the five white balls?

c. What is the probability of winning the Powerball jackpot?

Solution

a. $\frac{55!}{5!(55-5)!} = 3,478,761$

b. $\frac{1}{3,478,761}$

c. Number of choices $= (3,478,761)(42) = 146,107,962$

Probability of winning jackpot $= \frac{1}{146,107,962}$

This is one of the most important concepts in Probability. To illustrate it, let us look at an example.
A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

Definition

The **conditional probability** of an event **A**, given that an event **B** has occurred, is equal to

$$P(\mathbf{A}|\mathbf{B}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{B})},$$

provided $P(\mathbf{B}) > 0$. The symbol $P(\mathbf{A}|\mathbf{B})$ is read "probability of **A** given **B**".

Exercise 2.71

If two events, **A** and **B**, are such that $P(\mathbf{A}) = 0.5$, $P(\mathbf{B}) = 0.3$, and $P(\mathbf{A} \cap \mathbf{B}) = 0.1$, find the following:

a. $P(\mathbf{A}|\mathbf{B})$.
b. $P(\mathbf{B}|\mathbf{A})$.
c. $P(\mathbf{A}|\mathbf{A} \cup \mathbf{B})$.
d. $P(\mathbf{A}|\mathbf{A} \cap \mathbf{B})$.
e. $P(\mathbf{A} \cap \mathbf{B}|\mathbf{A} \cup \mathbf{B})$.

Solution

a. $P(\mathbf{A}|\mathbf{B}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{B})} = \frac{0.1}{0.3} = \frac{1}{3}$

b. $P(\mathbf{B}|\mathbf{A}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{A})} = \frac{1}{5}$

c. $P(\mathbf{A}|\mathbf{A} \cup \mathbf{B}) = \frac{P(\mathbf{A} \cap \mathbf{A} \cup \mathbf{B})}{P(\mathbf{A} \cup \mathbf{B})} = \frac{P(\mathbf{A})}{P(\mathbf{A} \cup \mathbf{B})} = \frac{5}{7}$

d. $P(\mathbf{A}|\mathbf{A} \cap \mathbf{B}) = 1$

e. $P(\mathbf{A} \cap \mathbf{B}|\mathbf{A} \cup \mathbf{B}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{A} \cup \mathbf{B})} = \frac{1}{7}$

Example

You are given the following table for a loss:

Amount of Loss	Probability
0	0.4
1	0.3
2	0.1
3	0.1
4	0.1

Given that the loss amount is positive, calculate the probability that it is more than 1.

Solution

Let $\mathbf{A}$ be the event that the loss amount is positive, and $\mathbf{B}$ the event that the loss amount exceeds 1. Then $\mathbf{A} \cap \mathbf{B}$ is clearly equal to $\mathbf{B}$ because the loss will necessarily be positive if it exceeds 1 (i.e., $\mathbf{B} \subset \mathbf{A}$).

The probability that the claim amount is positive (event $\mathbf{A}$) is $0.3 + 0.1 + 0.1 + 0.1 = 0.6$ and the probability that the claim amount is **greater** than 1 (event $\mathbf{B}$) is $0.1 + 0.1 + 0.1 = 0.3$.

The conditional probability is

$$P(\mathbf{B}|\mathbf{A}) = \frac{P(\mathbf{A} \cap \mathbf{B})}{P(\mathbf{A})} = \frac{P(\mathbf{B})}{P(\mathbf{A})} = \frac{0.3}{0.6} = 0.5$$

Two events **A** and **B** are called **independent** if one has no effect on the other. That means that whether **A** is given or not is irrelevant to $\mathbf{P}(\mathbf{B})$, i. e., $\mathbf{P}(\mathbf{B}|\mathbf{A}) = \mathbf{P}(\mathbf{B})$. It follows from the definition of conditional independence that

$$\mathbf{P}(\mathbf{A} \cap \mathbf{B}) = \mathbf{P}(\mathbf{A})\mathbf{P}(\mathbf{B})$$

Definition

Two events **A** and **B** are said to be **independent** if any one of the following holds:

$$\mathbf{P(A|B) = P(A)},$$

$$\mathbf{P(B|A) = P(B)},$$

$$\mathbf{P(A \cap B) = P(A)P(B)}.$$

Otherwise, the events are said to be **dependent**.

Exercise 2.79

Suppose that A and B are mutually exclusive events, with $0 < P(A) < 1$ and $0 < P(B) < 1$. Are A and B independent? Prove your answer.

Example

In a certain population, 60% own a car, 30% own a house and 20% own a house and a car. Determine whether or not the events that a person owns a car and that a person owns a house are independent.

Solution

Let **A** be the event that the person owns a car and **B** the event that the person owns a house. $P(\mathbf{A}) = 0.6$, $P(\mathbf{B}) = 0.3$ and $P(\mathbf{A} \cap \mathbf{B}) = 0.2 \neq 0.18 = P(\mathbf{A})P(\mathbf{B})$. Hence the two events are **not** independent.

Example

Workplace accidents are categorized as minor, moderate and severe. The probability that a given accident is minor is 0.5, that it is moderate is 0.4, and that it is severe is 0.1. Two accidents occur independently in one month. Calculate the probability that neither accident is severe and at most one is moderate.

Solution

Let us denote by L_1, M_1 and S_1 respectively the events that the first accident is minor (little), moderate and severe. Similarly for the second accident denote with 2 as a subscript. Since neither should be S and at most one should be M, the only possibilities are $L_1 \cap L_2$ or $L_1 \cap M_2$ or $M_1 \cap L_2$. Note that these possibilities are mutually exclusive. Since the two accidents are independent, the required probability is

$$P(L_1)P(L_2) + P(L_1)P(M_2) + P(M_1)P(L_2)$$

$$= (0.5)(0.5) + (0.5)(0.4) + (0.4)(0.5) = 0.65$$

Suppose that 80 percent of used car buyers are good credit risks. Suppose, further, that the probability is 0.7 that an individual who is a good credit risk has a credit card, but that this probability is only 0.4 for a bad credit risk. Calculate the probability

a. a randomly selected car buyer has a credit card.

b. a randomly selected car buyer who has a credit card is a good risk.

c. a randomly selected car buyer who does not have a credit card is a good risk.

Solution

a. G = selecting a good credit risk. Note that $P(G) = 0.8$.
Therefore, $P(G^c) = 0.2$
C = selecting an individual with a credit card.
$P(C|G) = 0.7$ and $P(C|G^c) = 0.4$.
$P(C) = P(C \cap G) + P(C \cap G^c) = P(G)P(C|G) + P(G^c)P(C|G^c)$
$P(C) = (0.8)(0.7) + (0.2)(0.4) = 0.64$

b. $P(G|C) = \frac{P(G \cap C)}{P(C)} = \frac{P(G)P(C|G)}{P(C)} = \frac{(0.8)(0.7)}{0.64} = \frac{7}{8}$

c. $P(G|C^c) = \frac{P(G \cap C^c)}{P(C^c)} = \frac{P(G)P(C^c|G)}{P(C^c)} = \frac{(0.8)(0.3)}{0.36} = \frac{2}{3}$

Example

A local bank reviewed its credit card policy with the intention of recalling some of its credit cards. In the past approximately 5% of cardholders defaulted, leaving the bank unable to collect the outstanding balance. Hence, management established a prior probability of 0.05 that any particular cardholder will default. The bank also found that the probability of missing a monthly payment is 20% for customers who do not default. Of course, the probability of missing a monthly payment for those who default is 1.

a. Given that a customer missed one or more monthly payments, compute the posterior probability that the customer will default.
b. The bank would like to recall its card if the probability that a customer will default is greater than 0.20. Should the bank recall its card if the customer misses a monthly payment? Why or why not?

Solution

D = Default, D^c = customer doesn't default, M = missed payment.

a. $P(D|M) = \frac{P(D \cap M)}{P(M)} = \frac{P(D \cap M)}{P(D \cap M) + P(D^c \cap M)}$.

$P(D) = 0.05 \quad P(D^c) = 0.95 \quad P(M|D^c) = 0.20 \; P(M|D) = 1$

$P(D|M) = \frac{(0.05)(1)}{(0.05)(1)+(0.95)(0.20)} = 0.2083$

b. Yes, the bank should recall its card if the customer misses a monthly payment.

The Multiplicative Law of Probability

The probability of the intersection of two events **A** and **B** is

$$P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A})P(\mathbf{B}|\mathbf{A}) = P(\mathbf{B})P(\mathbf{A}|\mathbf{B}).$$

If **A** and **B** are independent, then

$$P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A})P(\mathbf{B}).$$

Proof.
The multiplicative law follows directly from the definition of conditional probability.

Theorem 2.6

The Additive Law of Probability
The probability of the union of two events $\mathbf{A}$ and $\mathbf{B}$ is

$$P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A} \cap \mathbf{B}).$$

If $\mathbf{A}$ and $\mathbf{B}$ are mutually exclusive events, $P(\mathbf{A} \cap \mathbf{B}) = 0$ and

$$P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}).$$

Theorem 2.7

If **A** is an event, then

$$P(\mathbf{A}) = 1 - P(\mathbf{A}^c).$$

Exercise 2.85

If $\mathbf{A}$ and $\mathbf{B}$ are independent events, show that $\mathbf{A}$ and $\mathbf{B}^c$ are also independent. Are $\mathbf{A}^c$ and $\mathbf{B}^c$ independent?

Proof

$P(\mathbf{A} \cap \mathbf{B}) + P(\mathbf{A} \cap \mathbf{B}^c) = P(\mathbf{A})$
$P(\mathbf{A} \cap \mathbf{B}^c) = P(\mathbf{A}) - P(\mathbf{A} \cap \mathbf{B})$ (since **A** and **B** are independent)
$P(\mathbf{A} \cap \mathbf{B}^c) = P(\mathbf{A}) - P(\mathbf{A})P(\mathbf{B})$
$P(\mathbf{A} \cap \mathbf{B}^c) = P(\mathbf{A})[1 - P(\mathbf{B})]$
$P(\mathbf{A} \cap \mathbf{B}^c) = P(\mathbf{A})[P(\mathbf{B}^c)]$
Therefore, **A** and $\mathbf{B}^c$ are independent.

Exercise 2.95

Two events $\mathbf{A}$ and $\mathbf{B}$ are such that $P(\mathbf{A}) = 0.2$, $P(\mathbf{B}) = 0.3$, and $P(\mathbf{A} \cup \mathbf{B}) = 0.4$. Find the following:

a. $P(\mathbf{A} \cap \mathbf{B})$

b. $P(\mathbf{A^c} \cup \mathbf{B^c})$

c. $P(\mathbf{A^c} \cap \mathbf{B^c})$

d. $P(\mathbf{A^c} | \mathbf{B})$

Solution

a. $P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{A} \cup \mathbf{B}) = 0.1$
b. $P(\mathbf{A^c} \cup \mathbf{B^c}) = P(\mathbf{A} \cap \mathbf{B})^c = 1 - P(\mathbf{A} \cap \mathbf{B}) = 1 - 0.1 = 0.9$
c. $P(\mathbf{A^c} \cap \mathbf{B^c}) = P(\mathbf{A} \cup \mathbf{B})^c = 1 - P(\mathbf{A} \cup \mathbf{B}) = 1 - 0.4 = 0.6$
d. $P(\mathbf{A^c}|\mathbf{B}) = \frac{P(\mathbf{A^c} \cap \mathbf{B})}{P(\mathbf{B})} = \frac{0.2}{0.3} = \frac{2}{3}$
Note that $P(\mathbf{A^c}) = P(\mathbf{A^c} \cap \mathbf{B}) + P(\mathbf{A^c} \cap \mathbf{B^c})$.

Exercise 2.15

A football team has a probability of 0.75 of winning when playing any of the other four teams in its conference. If the games are independent, what is the probability the team wins all its conference games?

Solution

Let W_i denote the event our football team wins its ith conference game.
$P(W_1 \cap W_2 \cap W_3 \cap W_4) = P(W_1)P(W_2)P(W_3)P(W_4) = (0.75)^4 = \frac{81}{256} \approx 0.3164$

Example

Observation of a waiting line at a medical clinic indicates the probability that a new arrival will be an emergency case is $p = 1/6$. Find the probability that the rth patient is the first emergency case. (Assume that conditions of arriving patients represent independent events.)

Solution

The experiment consists of watching patient arrivals until the first emergency case appears. Then the sample points for the experiment are

E_i : The ith patient is the first emergency case, for $i = 1, 2, ...$

Because only one sample point falls in the event of interest,

$P(\text{rth patient is the first emergency case}) = P(E_r)$.

Now define A_i to denote the event that the ith arrival is **not** an emergency case. Then we can represent E_r as the intersection

$$E_r = A_1 \cap A_2 \cap A_3 \cap ... \cap A_{r-1} \cap A_r^c.$$

Applying the multiplicative law and because the events $A_1,\ A_2,\ A_3...,\ A_{r-1}$ and A_r^c are independent, it follows that

$$P(E_r) = P(A_1)P(A_2)...P(A_{r-1}) = (1-p)^{r-1}p$$

$$P(E_r) = (5/6)^{r-1}(1/6) \qquad r = 1, 2, 3, ...$$

A summary of the steps used in the event-composition method follows:

1. Define the experiment.

2. Visualize the nature of the sample points. Identify a few to clarify your thinking.

3. Write an equation expressing the event of interest - say, **A** - as a composition of two or more events, using unions, intersections, and/or complements.

4. Apply the additive and multiplicative laws of probability to the compositions obtained in step 3 to find $P(\mathbf{A})$.

Problem

Ten percent of a company's life insurance policyholders are smokers. The rest are nonsmokers. For each nonsmoker, the probability of dying during the year is 0.01. For each smoker, the probability of dying during the year is 0.05. Given that a policyholder has died, what is the probability that the policyholder was a smoker?

Solution

Let S = smoker, N = Nonsmoker, D = Dying, and Ω = sample space.
Note that $S \cup N = \Omega$ and $S \cap N = \emptyset$. We also have that $P(S) = 0.10$, $P(N) = 0.90$, $P(D|N) = 0.01$, and $P(D|S) = 0.05$.

$$P(S|D) = \frac{P(S \cap D)}{P(D)} = \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|N)P(N)}$$

$$P(S|D) = \frac{(0.05)(0.10)}{(0.05)(0.10) + (0.01)(0.90)} = \frac{5}{14}$$

A few comments about last problem

At first sight, our result sounds counterintuitive. Let us convince ourselves that it makes sense.

Assume that the company has 10,000 policyholders. Given the information that they have, 1000 of them are smokers and the remaining 9000 are nonsmokers. Furthermore, according to company's experience, 50 smokers are going to die during the year (0.05×1000). On the other hand, 90 nonsmokers are going to die during the year (0.01×9000). So, the company expects to have 140 deaths during the year.

Finally, note that the proportion of smokers who die is given by $\frac{50}{140} = \frac{5}{14}$. Doing a similar calculation, the proportion of nonsmokers who die is $\frac{90}{140} = \frac{9}{14}$.

Problem

The probability that a randomly chosen male has a circulation problem is 0.25. Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem. What is the conditional probability that a male has a circulation problem, given that he is a smoker?

Solution

Let C stand for circulation problem and S for smoker. We are given that $P(C) = 0.25$, and that $P(S|C) = 2P(S|C^c)$. We need to find $P(C|S)$.

$$P(C|S) = \frac{P(S|C)P(C)}{P(S|C)P(C) + P(S|C^c)P(C^c)}$$

$$P(C|S) = \frac{P(S|C)(0.25)}{P(S|C)(0.25) + (1/2)P(S|C)(0.75)} = 0.4.$$

Problem

An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	0.15
Young adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

Solution

Y = Young adult and C = collision. We have that
$P(C|Y)P(Y) = (0.16)(0.08)$, $P(C|T)P(T) = (0.15)(0.08)$,
$P(C|M)P(M) = (0.04)(0.45)$ and $P(C|S)P(S) = (0.05)(0.31)$.

$$P(Y|C) = \frac{(0.08)(0.16)}{(0.15)(0.08) + (0.08)(0.16) + (0.04)(0.45) + (0.05)(0.31)}$$

$$P(Y|C) = 0.21955$$

Example

An insurance company classifies drivers as High Risk, Standard and Preferred. 10% of the drivers in a population are High Risk, 60% are Standard and the rest are preferred. The probability of accident during a period is 0.3 for a High Risk driver, 0.2 for a Standard driver and 0.1 for a preferred driver.

1. Given that a person chosen at random had an accident during this period, find the probability that the person is Standard.
2. Given that a person chosen at random has not had an accident during this period, find the probability that the person is High Risk.

Solution

Let H, S, P and A stand for High-Risk, Standard, Preferred and Accident and W stand for H, S or P.

	$Pr(W)$	$Pr(A\|W)$	$Pr(A \cap W) = Pr(A\|W)Pr(W)$
H	0.10	0.30	0.03
S	0.60	0.20	0.12
P	0.30	0.10	0.03
Total	1		

Solution (1)

1. From Bayes' Formula, taking the figures from the table,

$$Pr(S|A) = \frac{Pr(A|S)Pr(S)}{Pr(A|S)Pr(S) + Pr(A|H)Pr(H) + Pr(A|P)Pr(P)}$$

$$Pr(S|A) = \frac{0.12}{0.18} = \frac{2}{3}$$

Solution (2)

2. For the second part, note that $Pr(A^c|W) = 1 - Pr(A|W)$. This is because if a fraction of p amongst W get into an accident, then the fraction $1 - p$ of W does not get into an accident. You can draw another table or observe that

$$Pr(H|A^c) = \frac{Pr(A^c|H)Pr(H)}{Pr(A^c)} = \frac{(1-0.3)(0.1)}{1-0.18} = \frac{0.07}{0.82} = \frac{7}{82}.$$

Definition

For some positive integer k, let the sets $B_1, B_2, ..., B_k$ be such that
1.$\mathbf{S} = \mathbf{B}_1 \cup \mathbf{B}_2 \cup ... \cup \mathbf{B}_k$.
2.$\mathbf{B}_i \cap \mathbf{B}_j = \emptyset$, for $i \neq j$.
Then the collection of sets $\{B_1, B_2, ..., B_k\}$ is said to be a partition of $\mathbf{S}$.

Theorem 2.8

Assume that $\{B_1, B_2, ..., B_k\}$ is a partition of $\mathbf{S}$ such that $P(\mathbf{B}_i) > 0$, for $i = 1, 2, ..., k$. Then for any event $\mathbf{A}$

$$P(\mathbf{A}) = \sum_{i=1}^{k} P(\mathbf{A}|\mathbf{B}_i)P(\mathbf{B}_i)$$

Bayes' Rule

Assume that $\{B_1, B_2, ..., B_k\}$ is a partition of **S** such that $P(\mathbf{B}_i) > 0$, for $i = 1, 2, ..., k$. Then

$$P(\mathbf{B}_j|\mathbf{A}) = \frac{P(\mathbf{A}|\mathbf{B}_j)P(\mathbf{B}_j)}{\sum_{i=1}^{k} P(\mathbf{A}|\mathbf{B}_i)P(\mathbf{B}_i)}.$$

Definition

A **random variable** is a real-valued function for which the domain is a sample space.

Example

A fair die is thrown twice. The sample points are $(1,1)$, $(1,2)$, $(1,3), ...,$ $(6,6)$. There are 36 sample points. Let us assign the same probability $1/36$ for each of these points. Suppose we are interested in the sum of the numbers of each outcome. Then we can define a random variable $X = i + j$ associated with the outcome (i,j). List the possible values of X and say what the probability of each value is.

Solution

Value of X	Probability
2	$1/36$
3	$2/36$
4	$3/36$
5	$4/36$
6	$5/36$
7	$6/36$
8	$5/36$
9	$4/36$
10	$3/36$
11	$2/36$
12	$1/36$

Let N and n represent the numbers of elements in the population and sample, respectively. If the sampling is conducted in such a way that each of the $\frac{N!}{n!(N-n)!}$ samples has an equal probability of being selected, the sampling is said to be random, and the result is said to be a **random sample**.