STA215
Inference about comparing two populations

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- The goal of inference is to compare the responses to two treatments or to compare the characteristics of two populations.
- We have a separate sample from each treatment or each population.

Large-sample confidence interval for comparing two proportions

Draw an SRS of size n_1 from a population having proportion p_1 of successes and draw an independent SRS of size n_2 from another population having proportion p_2 of successes. When n_1 and n_2 are large, an approximate level C confidence interval for $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) \pm z^* SE$$

In this formula the standard error SE of $\hat{p}_1 - \hat{p}_2$ is

$$SE = \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}$$

and z^* is the critical value for the standard Normal density curve with area C between $-z^*$ and z^*.

Example: How to quit smoking

Nicotine patches are often used to help smokers quit. Does giving medicine to fight depression help? A randomized double-blind experiment assigned 244 smokers who wanted to stop to receive nicotine patches and another 245 to receive both a patch and the antidepression drug bupropion. Results: After a year, 40 subjects in the nicotine patch group had abstained from smoking, as had 87 in the patch-plus-drug group. Give a 99% confidence interval for the difference (treatment minus control) in the proportion of smokers who quit.

Solution

Let $\hat{p}_2$ = proportion of smokers who quit (control group) and $\hat{p}_1$ = proportion of smokers who quit (treatment group).
$\hat{p}_2 = \frac{40}{244} \approx 0.1639$ and $\hat{p}_1 = \frac{87}{245} \approx 0.3551$.
The standard error is
$SE = \sqrt{\frac{(0.1639)(1-0.1639)}{244} + \frac{(0.3551)(1-0.3551)}{245}} \approx 0.0387$.
The 99% confidence interval is:

$$(0.3551 - 0.1639) \pm 2.576(0.0387)$$

Lower Confidence Limit $= 0.1912 - 0.0996 = 0.0915$
Upper Confidence Limit $= 0.1912 + 0.0996 = 0.2908$

R Code

```
successes=c(87, 40);

totals=c(245, 244);

prop.test(successes,totals, conf.level=0.99,
correct=FALSE);
```

```
##
##  2-sample test for equality of proportions without conti
##  correction
##
## data:  successes out of totals
## X-squared = 23.237, df = 1, p-value = 1.432e-06
## alternative hypothesis: two.sided
## 99 percent confidence interval:
##  0.09152484 0.29081039
## sample estimates:
##    prop 1    prop 2
## 0.3551020 0.1639344
```

```
successes=c(40,87);

totals=c(244,245);

prop.test(successes,totals, conf.level=0.99,
correct=FALSE);
```

```
##
##  2-sample test for equality of proportions without conti
##  correction
##
## data:  successes out of totals
## X-squared = 23.237, df = 1, p-value = 1.432e-06
## alternative hypothesis: two.sided
## 99 percent confidence interval:
##  -0.29081039 -0.09152484
## sample estimates:
##    prop 1    prop 2
## 0.1639344 0.3551020
```

To test the hypothesis $H_0 : p_1 = p_2$ first find the pooled proportion $\hat{p}$ of successes in both samples combined ($\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$). Then compute the z_* statistic, $z_* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$

In terms of a variable Z having the standard Normal distribution, the approximate P-value for a test of H_0 against

$H_a : p_1 > p_2$: is : $P(Z > z_*)$

$H_a : p_1 < p_2$: is : $P(Z < z_*)$

$H_a : p_1 \neq p_2$: is : $2P(Z > |z_*|)$

Use this test when the populations are at least 10 times as large as the samples and the counts of successes and failures are each 10 or more in both samples.

How significant is the evidence that the medicine increases the success rate? State hypotheses, calculate a test statistic, use Table A to give its P-value, and state your conclusion. (Use $\alpha = 0.01$)

Let p_1 = proportion of ALL smokers who quit (control group) and p_2 = proportion of ALL smokers who quit (treatment group).

Step 1. State Hypotheses. $H_0 : p_1 = p_2$ vs $H_a : p_1 < p_2$

Step 2. Find test statistic. $\hat{p}_1 = \frac{40}{244} = 0.1639$ and $\hat{p}_2 = \frac{87}{245} = 0.3551$

$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{40 + 87}{244 + 245} = 0.2597$

$z_* = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = -4.82$

Step 3. Compute P-value.
$P - value = P(Z < z^*) = P(Z < -4.82) < 0.0003$

Step 4. Conclusion. Since $P - value < 0.0003 < \alpha = 0.01$, we **reject** the null hypothesis that $p_1 = p_2$. The data provide very strong evidence that bupropion increases success rate.

R Code

```
successes=c(87, 40);

totals=c(245, 244);

prop.test(successes,totals, alternative="greater",
correct=FALSE);
```

R Code

```
##
##  2-sample test for equality of proportions without conti
##  correction
##
## data:  successes out of totals
## X-squared = 23.237, df = 1, p-value = 7.161e-07
## alternative hypothesis: greater
## 95 percent confidence interval:
##  0.1275385 1.0000000
## sample estimates:
##    prop 1    prop 2
## 0.3551020 0.1639344
```

A hospital administrator suspects that the delinquency rate in the payment of hospital bills has increased over the past year. Hospital records show that the bills of 48 of 1284 persons admitted in the month of April have been delinquent for more than 90 days. This number compares with 34 of 1002 persons admitted during the same month one year ago. Do these data provide sufficient evidence to indicate an increase in the rate of delinquency in payments exceeding 90 days? Test using $\alpha = 0.10$.

Let p_1 and p_2 represent the proportions of all potential hospital admissions in April of this year and last year, respectively, that would have allowed their accounts to be delinquent for a period exceeding 90 days, and let $n_1 = 1284$ admissions this year and the $n_2 = 1002$ admissions last year represent independent random samples from these populations.

Step 1. State Hypotheses. $H_0 : p_1 = p_2$ vs $H_a : p_1 > p_2$

Step 2. Find test statistic. $\hat{p}_1 = \frac{48}{1284} = 0.0374$ and
$\hat{p}_2 = \frac{34}{1002} = 0.0339$

$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{48 + 34}{1284 + 1002} = 0.0359$

$z_* = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = 0.45$

Step 3. Compute P-value.
$P - value = P(Z > z^*) = P(Z > 0.45) = 1 - P(Z < 0.45) = 0.3264$

Step 4. Conclusion. Since $P - value > \alpha = 0.10$, we **cannot** reject the null hypothesis that $p_1 = p_2$. The data present insufficient evidence to indicate that the proportion of delinquent accounts in April of this year exceeds the corresponding proportion last year.

```
successes=c(48,34);

totals=c(1284,1002);

prop.test(successes,totals,alternative="greater",
correct=FALSE);
```

R Code

```
## 
##  2-sample test for equality of proportions without conti
##  correction
## 
## data:  successes out of totals
## X-squared = 0.19381, df = 1, p-value = 0.3299
## alternative hypothesis: greater
## 95 percent confidence interval:
##  -0.009368431  1.000000000
## sample estimates:
##     prop 1     prop 2
## 0.03738318 0.03393214
```

These statistics were calculated from two random samples:
$\hat{p}_1 = 0.60$ $n_1 = 225$ $\hat{p}_2 = 0.56$ $n_2 = 225$.
Calculate the P-value of a test to determine whether there is
evidence to infer that the population proportions differ.

After sampling from two binomial populations we found the
following.
$\hat{p}_1 = 0.18$ $n_1 = 100$ $\hat{p}_2 = 0.22$ $n_2 = 100$.
Estimate with 90% confidence the difference in population
proportions.

The National Sleep Foundation asked a random sample of 1010 U.S. adults questions about their sleep habits. The sample was selected in the fall of 2001 from random telephone numbers, stratified by region and sex, guaranteeing that an equal number of women and men were interviewed.

One of the questions asked about snoring. Of the 995 respondents, 37% of adults reported that they snored at least a few nights a week during the past year. Would you expect that percentage to be the same for all age groups? Split into two age categories, 26% of the 184 people under 30 snored, compared with 39% of the 811 in the older group. Is this difference of 13% real, or due only to natural fluctuations in the sample we've chosen? (Use $\alpha = 0.01$).

H_0: There is no difference in snoring rates in the two age groups.

H_a: The rates are different.

- Independence assumption: The National Sleep Foundation elected respondents at random, so they should be independent.
- 10% condition: The number of adults surveyed is certainly far less than 10% of the population.
- Independent groups assumptions: The two groups are independent of each other because the sample was selected at random.
- Success/Failure condition: In the younger age group, 48 snored and 136 didn't. In the older group, 318 snored and 493 didn't. The observed numbers of both successes and failures are much more than 10 for both groups.

Let $p_1 = p_{old}$ and $p_2 = p_{young}$.

Step 1. State Hypotheses. $H_0 : p_1 = p_2$ vs $H_a : p_1 \neq p_2$

Step 2. Find test statistic. $\hat{p}_1 = \frac{318}{811} = 0.392$ and $\hat{p}_2 = \frac{48}{184} = 0.261$

$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{318 + 48}{811 + 184} = 0.3678$

$z_* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.131}{0.0394} \approx 3.33$

Step 3. Compute P-value.
$P-value = 2P(Z > |z^*|) = 2P(Z > 3.33) = 2[1 - P(Z < 3.33)] < 2[1 - 0.9996] = 0.0008$

Step 4. Conclusion. Since $P-value = 0.0008 < \alpha = 0.01$, we **reject the null hypothesis** that $p_1 = p_2$. The P-value says that if there really were no difference in (reported) snoring rates between the two age groups, then the difference observed in this study would happen only 8 times in 10,000. This is so small that we reject the null hypothesis of no difference and conclude that there is a difference in the rate of snoring between older adults and younger adults. It appears that older adults are more likely to snore.

One hundred normal-weight people and 100 obese people were observed at several Chinese-food buffets. For each researchers recorded whether the diner used chopsticks or knife and fork. The table shown here was created.

	Normal Weight	Obese
Used chop sticks	26	7
Used knife and fork	74	93

Is there sufficient evidence at the 10% significance level to conclude that obese Chinese food eaters are less likely to use chop sticks?

Let p_1 represent the proportion of all Normal Weight Chinese food eaters that use chop sticks and p_2 represent the proportion of all obese Chinese food eaters that use chop sticks.

Step 1. State Hypotheses. $H_0 : p_1 = p_2$ vs $H_a : p_1 > p_2$

Step 2. Find test statistic. $\hat{p}_1 = \frac{26}{100} = 0.26$ and $\hat{p}_2 = \frac{7}{100} = 0.07$

$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{26 + 7}{100 + 100} = 0.165$

$z_* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = 3.6195 \approx 3.62$

Step 3. Compute P-value.
$P - value = P(Z > z^*) = P(Z > 3.62) = 1 - P(Z < 3.62) <$
$1 - 0.9998 = 0.0002$ (you can find the exact P-value using R).

Step 4. Conclusion. Since $P - value < 0.0002 < \alpha = 0.10$, we
reject the null hypothesis that $p_1 = p_2$. There is enough
evidence to conclude that obese Chinese food eaters are less likely
to use chop sticks.