CSC358 Tutorial 9

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Question 1: Concept Review

Consider the following four desirable characteristics of a broadcast channel. Which of these characteristics are satisfied by FDMA, pure ALOHA, slotted ALOHA, and CSMA? Let R be the bandwidth of channel.

(a) When only one node has data to send, that node has a throughput of R.

(b) When M node have data to send, each of these nodes, on average, has a fair share of the channel bandwidth.

(c) The protocol is decentralized, i.e., there is no master node that represents a single point of failure.

(d) The protocol is simple, so that it is inexpensive to implement

For the answers, review the lectures, books, go to office hours, and use the discussion board!

Question 2: Pure ALOHA: Proof for Efficiency

In the lecture, we performed a proof that the maximum of slotted ALOHA is $\frac{1}{e}$ when $N \to \infty$.

In this question, carry out a similar proof for pure ALOHA, i.e., show that its maximum efficiency is $\frac{1}{2 \cdot e}$.

Write down all steps of your derivation carefully.

Question 2: Pure ALOHA: Proof for Efficiency

As a reminder, in order to prove pure ALOHA's maximum efficiency, we must:

Step 1: Take the derivative of efficiency formula: $N \cdot p(1-p)^{2\cdot N-2}$

Step 2: Set to 0 and find p^*

Step 3: Plug p^* back into the efficiency formula and let N go to infinity

Question 2: Pure ALOHA: Proof for Efficiency

Step 1: Take the derivative of efficiency formula: $N \cdot p(1-p)^{2\cdot N-2}$

$$
\begin{aligned}
\frac{\partial}{\partial p}\left(N \cdot p(1-p)^{2\cdot N-2}\right) &= N\left(\frac{\partial}{\partial p}\left(p(1-p)^{2\cdot N-2}\right)\right)\\
&= N\left(p\left(\frac{\partial}{\partial p}\left((1-p)^{2\cdot N-2}\right)\right) + (1-p)^{2\cdot N-2}\left(\frac{\partial}{\partial p}(p)\right)\right)\\
&= N\Bigg(p(2\cdot N-2)(1-p)^{2\cdot N-3}\left(\frac{\partial}{\partial p}(1-p)\right)\\
&\qquad + (1-p)^{2\cdot N-2}\left(\frac{\partial}{\partial p}(p)\right)\Bigg)\\
&= N\Bigg(p(2\cdot N-2)(1-p)^{2\cdot N-3}\left(-\frac{\partial}{\partial p}(p)\right)\\
&\qquad + (1-p)^{2\cdot N-2}\left(\frac{\partial}{\partial p}(p)\right)\Bigg)\\
&= N\left(-p(2\cdot N-2)(1-p)^{2\cdot N-3} + (1-p)^{2\cdot N-2}\right)\\
&= N(1-p)^{2\cdot N-3}\left(-2\cdot p(\cdot N-1) + (1-p)\right)
\end{aligned}
$$

Step 2: Set to 0 and find p^*

$$
\begin{aligned}
0 &= N(1-p)^{2\cdot N-3}(-p(2\cdot N-1)+(1-p)) \\
\implies 0 &= -p(2\cdot N-1)+(1-p) \\
\implies \frac{p-1}{p} &= -(2\cdot N-1) \\
\implies \frac{1}{p} &= 2\cdot N-2+1 \\
\implies p &= \frac{1}{2\cdot N-1}
\end{aligned}
$$

Hence we get $p^* = \frac{1}{2\cdot N-1}$

Step 3: Plug p^* back into the efficiency formula and let N go to infinity

Once we plug in p^* to the efficiency formula, we get:

$$\frac{N}{2 \cdot N - 1}(1 - \frac{1}{2 \cdot N - 1})^{2 \cdot N - 2}$$

Now, if we let N go to infinity:

$$\lim_{N \to \infty} \frac{N}{2 \cdot N - 1}(1 - \frac{1}{2 \cdot N - 1})^{2 \cdot N - 2} = \frac{1}{2} \cdot \frac{1}{e} = \frac{1}{2 \cdot e}$$

Hence, we have proved that the maximum efficiency for pure ALOHA is $\frac{1}{2 \cdot e}$

Consider two nodes, A and B, that use the slotted ALOHA protocol to contend for a channel with bandwidth R. Suppose node A has more data to transit than node B, and node A's retransmission probability p_A is greater than node B's retransmission probability, p_B.

(a) Provide a formula for node A's average throughput.

(b) Provide a formula for node B's average throughput.

(c) What is the total efficiency of the protocol with these two nodes?

(d) If $p_A = 2 \cdot p_B$, is node A's average throughput twice as large as that of node B? Why or why not? If not, how can you choose p_A and p_B to make that happen?

(e) In general, suppose there are N nodes, among which node A has retransmission probability $2 \cdot p$ and all other nodes have retransmission probability p. Provide expressions to compute the average throughputs of node A and any other node.

Question 3: Slotted ALOHA: Performance Analysis

(a) A's average throughput is given by $p_A(1 - p_B)$

(b) B's average throughput is given by $p_B(1 - p_A)$

(c) Total efficiency is:
A's avg throughput + B's avg throughput $= p_A(1 - p_B) + p_B(1 - p_A)$

(d) If $p_A = 2 \cdot p_B$, then the average throughput of A becomes: $2 \cdot p_B(1 - p_B)$.
Furthermore, the average throughput of B becomes: $p_B(1 - 2 \cdot p_B)$.
We can clearly see that throughput of A is not twice as large as B's.

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We can clearly see that throughput of A is not twice as large as B's.
In order to make throughput of A double that of B's, the following must hold:

$$p_A(1 - p_B) = 2\,(p_B(1 - p_A))$$

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Furthermore, the average throughput of B becomes: $p_B(1 - 2 \cdot p_B)$.
We can clearly see that throughput of A is not twice as large as B's.
In order to make throughput of A double that of B's, the following must hold:

$$p_A(1 - p_B) = 2\left(p_B(1 - p_A)\right)$$

Hence, we require that $p_A = \frac{2 \cdot p_B}{1 + p_B}$

(e) Given that A's retransmission probability is $2 \cdot p$, and all other $(N-1)$ nodes have retransmission probability of p, the average throughput of A must be:

$$2 \cdot p(1-p)^{N-1}$$

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For the rest of the nodes, the average throughput will be:

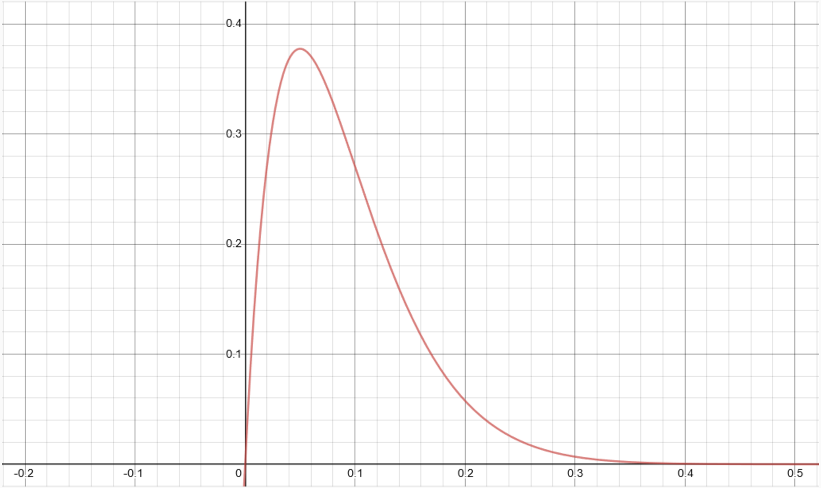
$$p(1-p)^{N-2}(1-2 \cdot p)$$

Question 4: Visualization

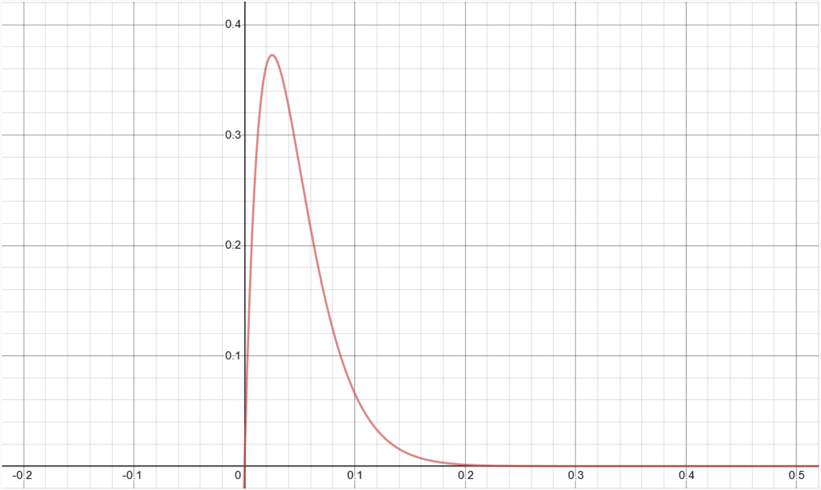
Graph the efficiency of slotted ALOHA and pure ALOHA as a function of p for $N = 20, 40, 60$ (using whichever plotting tool). Compare them and explain their differences.

Question 4: Slotted ALOHA

Here is a graph of slotted ALOHA's efficiency, with $N = 20$. The x-axis is the probability of transmission, the y-axis is the resulting efficiency (the probability that a time slot won't be wasted)



For $N = 40$:



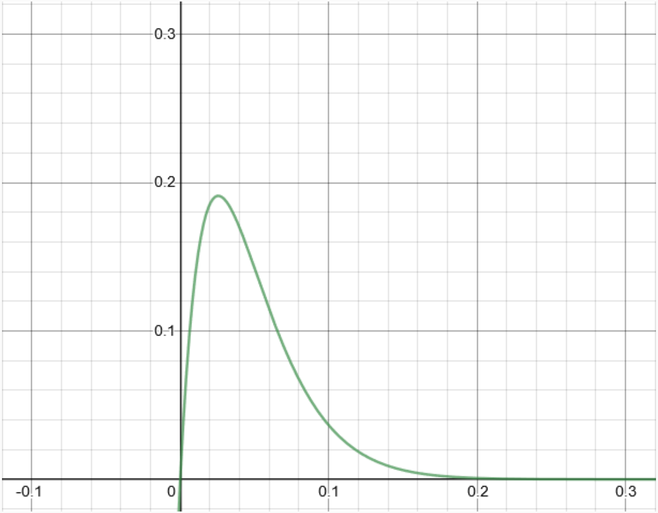
For $N = 60$:

Every time we increased the number of nodes, the curve became skinnier proportionally. Intuitively, it's harder to avoid collisions when there are more nodes. But the peak efficiency remains roughly the same, at around 0.3774, when $p = 1/N$.

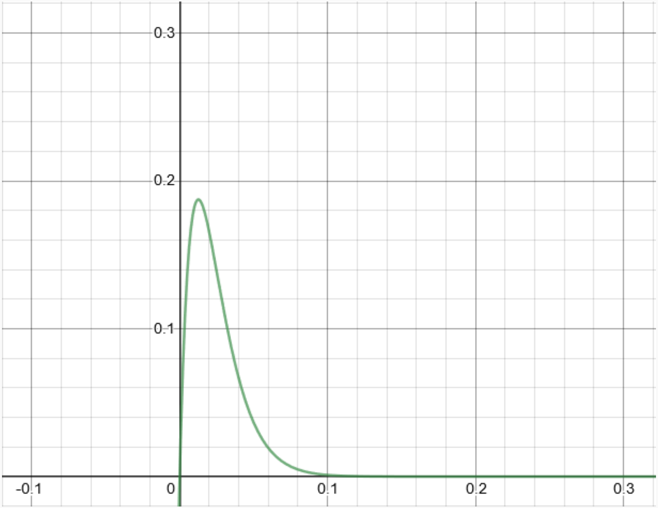
The same trend applies to plots of Pure ALOHA, except in this case the peak efficiency is worse at around 0.1863

Question 4: Pure ALOHA

Pure ALOHA for $N = 20$:



Pure ALOHA for $N = 40$:



Pure ALOHA for $N = 60$: