CSC338 Winter 2022

Week 9 - Newton's Method

Ilir Dema

University of Toronto

Mar 18, 2022

Recap of Interval Bisection Method

Interval Bisection

- Guaranteed convergence
- Rate of convergence is independent of the function
- Only evaluate the function (not the derivative)
- Slow to converge (linear)
- Does not use much info about f, just its sign at the extremities of the search interval
- Requires initial bounds for the root

Recap of Fixed Point Method

Fixed Point

- Rewrite your equation as $g(x) = x$
- Find fixed point of $g(x)$.
- If $g(x)$ has a fixed point x^* and $|g'(x^*)| < 1$ then there exist an interval about x^* such that if initial value x_0 is in this interval, then the fixed point iteration $x_{k+1} = g(x_k)$ converges to x^*.
- Downside: not so easy to decide the form $g(x) = x$ (Q: Explain, why is this a downside?)
- Example: $x^3 - x - 1 = 0$
    - $g(x) = x^3 - 1$ diverges
    - $g(x) = (x+1)^{1/3}$ converges
- Q: What happens if $|g'(x^*)| \approx 1$?

NEWTON'S METHOD IDEA

Newton's method approximates nonlinear function f near x_k by tangent line at $f(x_k)$

If x^* is the root, we can approx. it with x_{k+1}.

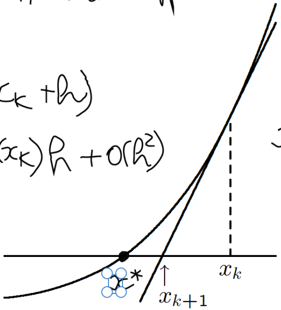
Hence:

$$0 = f(x_{k+1}) = f(x_k + h)$$

$$= f(x_k) + f'(x_k)h + O(h^2)$$

$$h = x_{k+1} - x_k$$

Ignore h.o. term $O(h^2)$.



Solve:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Newton's Method

The math behind

- If we take the tangent at a point nearby the root, the tangent and the curve intersect x-axes approximatley at the same point.
- Hence,using Taylor's expansion:
  $f(x_{k+1}) = f(x_k) + f'(x_k)(x_{k+1} - x_k) + \frac{1}{2}f''(\xi)(x_{k+1} - x_k)^2$
  $\xi \in [x_{k+1}, x_k]$ and f is smooth, so the error term $\frac{1}{2}f''(\xi)(x_{k+1} - x_k)^2$ can be ignored.
- The the linear approximation is $f(x_{k+1}) \approx f(x_k) + f'(x_k)(x_{k+1} - x_k)$ which can be solved for x_{k+1}, assuming $f(x_{k+1}) \approx 0$.
- x_0 need be chosen wisely!

Newton's Method Example

Example: Newton's Method

Use Newton's method to find root of

$$f(x) = x^2 - 4\sin(x) = 0$$

Derivative is

$$f'(x) = 2x - 4\cos(x),$$

so iteration scheme is

$$x_{k+1} = x_k - \frac{x_k^2 - 4\sin(x_k)}{2x_k - 4\cos(x_k)}$$

Taking $x_0 = 3$ as starting value, we obtain

x	$f(x)$	$f'(x)$	h
3.000000	8.435520	9.959970	-0.846942
2.153058	1.294772	6.505771	-0.199019
1.954039	0.108438	5.403795	-0.020067
1.933972	0.001152	5.288919	-0.000218
1.933754	0.000000	5.287670	0.000000

Newton's Method Covergence

Convergence

- Letting $g(x) = x - \frac{f(x)}{f'(x)}$ we see Newton's method is equivalent to the computation of fixed point for $g(x)$.
- Since $g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$ at a root x^* of f, we have $g'(x^*) = 0$.
- Continuity of g' implies $|g'(x)| < 1$ in $(x^* - \delta, x^* + \delta)$. $g(x) = g(x^*) + g'(x^*)(x - x^*) + \frac{1}{2}g''(\xi)(x - x^*)^2$, note $g(x^*) = x^*, g'(x^*) = 0$, hence $x_{k+1} - x_k = \frac{1}{2}g''(\xi)(x_k - x^*)^2$
- g'' is continuous, hence $|g''|$ is bounded in $(x^* - \delta, x^* + \delta)$.
- Therefore, $|x_{k+1} - x_k| \leq M|x_k - x^*|^2$.
- This shows x_k converges quadratically at x^*.

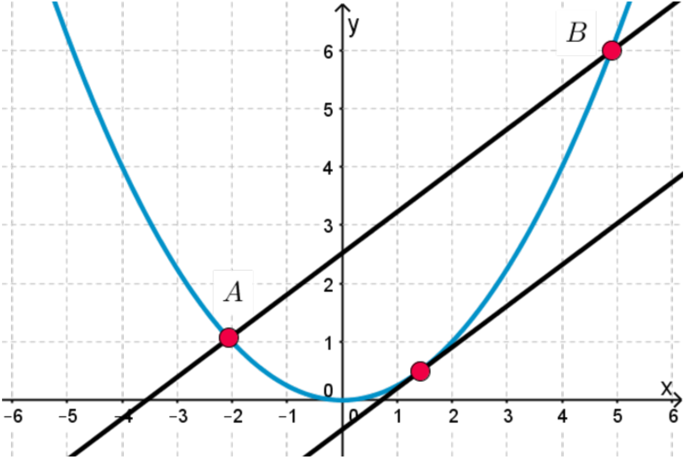
Multiple Roots Case

For multiple root, Newton's method linearly convergent (with constant $C = 1-(1/m)$, where m is multiplicity)

Both cases illustrated below

k	$f(x) = x^2 - 1$	$f(x) = x^2 - 2x + 1$
0	2.0	2.0
1	1.25	1.5
2	1.025	1.25
3	1.0003	1.125
4	1.00000005	1.0625
5	1.0	1.03125

THE IDEA OF THE SECANT



Both lines have same slope, hence computing secant slope saves us the trouble of computing the derivative.

The secant method

The method and its convergence

- Aproximate the derivative using the finite difference $f'(x_k) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$.
- Some algebra gives $x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1}}$
- Convergence rate of the secant method is approx 1.62, so it is superliner, but not quite quadratic. (Proof in the lecture).

REFERENCES

Michael T. Heath
Scientific Computing (Revised Second Edition)
SIAM