CSC338 Winter 2022

Week 8 - Nonlinear equations

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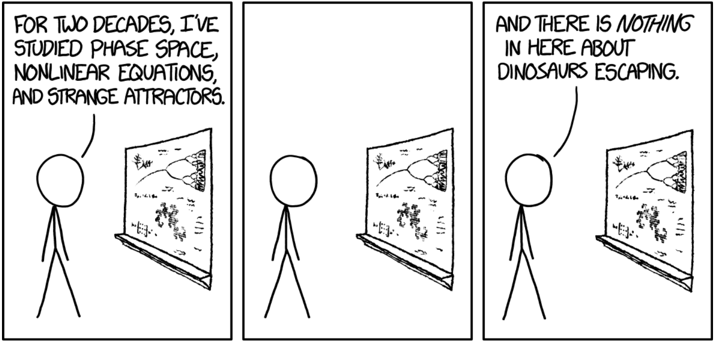
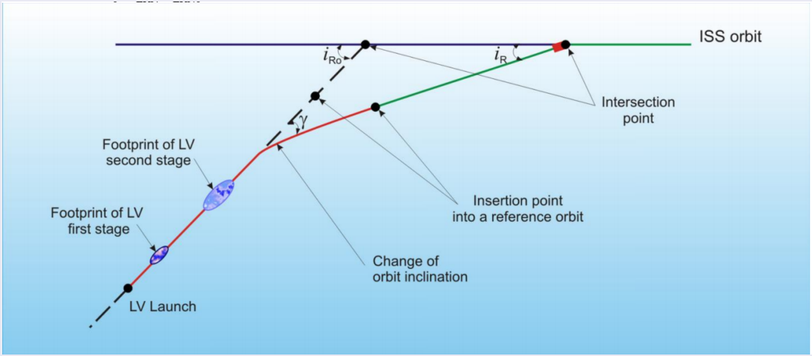


https://xkcd.com/1399/

Solving nonlinear equations is ...

... rocket science!



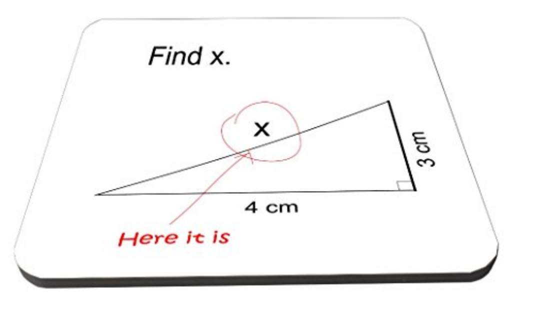
Reference: Quasy-coplanar instertion to implement quick two-orbit rendezvous profile of soyz spacecraft.
Dr. Rafail Murtazin

Nonlinear equations

What is a nonlinear equation?

Simply put, a nonlinear equation is an equation of the form $f(\boldsymbol{x}) = 0$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function (smooth means 'has derivatives of all orders').

Example: $x^2 - 3^2 - 4^2 = 0$



What is a solution to a nonlinear equations?

Definition

Root (solution) of a nonlinear equation $f(\boldsymbol{x}) = 0$ is called the vector $\boldsymbol{x}^* \in \mathbb{R}^n$ such that $f(\boldsymbol{x}^*) = 0$.

Examples

- $e^x + x^2 = 0$, in one dimension, has no roots
- $x + \cos x = 0$, in one dimension, has a single root $x \approx -0.739085$
- In two dimensions, one root, $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$:

  $$\begin{aligned} x_1 x_2 - 2x_2 + 1 &= 0 \\ e^{x_2 - 1} - 1 &= 0 \end{aligned}$$

- $\sin x = 0$, in one dimension, an infinity of roots: $x = n\pi, n \in \mathbb{Z}$.

The math behind nonlinear equations

Facts

- Often no formulas do exist for solving nonlinear equations ($3x^5 - 10x^4 - 10x^3 + 60x^2 + 95x - 100 = 0$, it took a few hundred years to understand that)
- So, often, numerical methods are the only way to solve them!

Relevant theorem

- Bolzano (corollary to intermediate value theorem): if $f : [a,b] \to \mathbb{R}$ is continuous and $f(a)f(b) < 0$ then there exists $c \in (a,b)$ such that $f(c) = 0$.
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Existence and unicity

Inverse function theorem

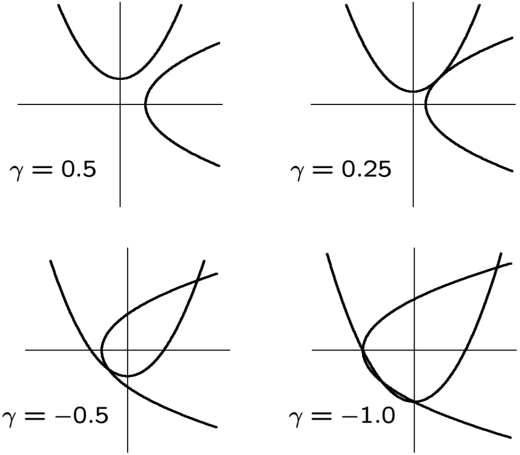
- If $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continunously differentiable, and around a point $\boldsymbol{c}$ its Jacobian $\boldsymbol{J_f}(c) = \left[\frac{\partial f_i}{\partial x_j}(\boldsymbol{c})\right]$ is non-degenrate, then $\boldsymbol{f}$ is invertible and there exists a neighborhood of $\boldsymbol{f(c)}$ such that the equation $\boldsymbol{f(x) = y}$ has a unique solution for any $\boldsymbol{y}$ in that neighborhood.
- Please note, more often than not, this is more of a theoretical tool.

Multiple roots

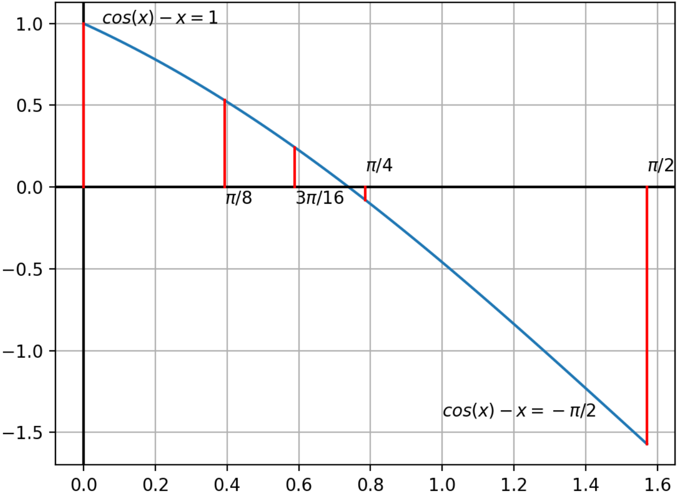
If the above does not hold, the equation may have multiple roots, for example $x^2 - 2x + 1 = 0$ has $x = 1$ a multiple root (check the derivative of $f(x) = x^2 - 2x + 1$ at $x = 1$!)

Systems of equations depending on a parameter

$$\begin{aligned} x_1^2 - x_2 + \gamma &= 0 \\ -x_1 + x_2^2 + \gamma &= 0 \end{aligned}$$



Interval Bisection Method

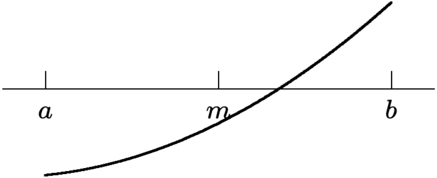


Interval Bisection Method

Interval Bisection Method

Bisection method begins with initial bracket and repeatedly halves its length until solution has been isolated as accurately as desired

while $((b-a) > tol)$ **do**
  $m = a + (b-a)/2$
  if $\text{sign}(f(a)) = \text{sign}(f(m))$ **then**
    $a = m$
  else
    $b = m$
  end
end



Pitfalls of Interval Bisection Method

Floating point arithmetic

Q: Why do we compute $m = a + (b-a)/2$ and not $m = (a+b)/2$?

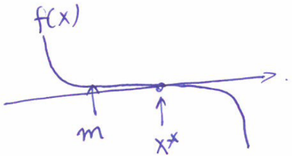
Example

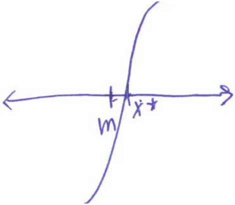
Consider $\mathcal{F}(\beta = 10, p = 2, U = -10, L = 10)$. Let $a = 6.7 \times 10^{-1}, b = 6.9 \times 10^{-1}$. Then $a+b = 1.36$ which rounds to 1.4×10^{0}. Next, $(a+b)/2 = 0.7 = 7 \times 10^{-1}$ is outside $[a,b]$. On the other end, $m = a + (b-a)/2$ is always guaranteed to be in $[a,b]$.

Stopping Criterion

Depends on the case

- When the search interval is small enough: $|m - x^*| < \varepsilon$
  - good when interval is small for $f(x^*)$ large
- When value of $f(x^*) < \varepsilon$ smaller then a preselected threshold.
  - good when interval is large but $f(x^*)$ is small





Conditioning

Assume f is differentiable. Then:

- $C_N = \frac{|f(x+\Delta x) - f(x)|/|f(x)|}{|\Delta x|/|x|} \approx \frac{|xf'(x)|}{f(x)}$
- Please note this definition is not good when $f(x) = 0$ (root finding).
- In such cases, C_N can be defined as the ratio of absolute errors, leading to
- $C_N = 1/|f'(x)|$ (why?) which means the root finding problem is well conditioned if the slope at the root is $>> 1$.

Convergence rates

Definitions

- A sequence a_n converges to zero if $\lim_{n\to\infty} a_n = 0$.
$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n > N \implies |a_n| < \varepsilon.$$
- Let a_n a sequence with nonzero terms. If there exists $C, r \in \mathbb{R}, C > 0, r > 0$ such that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|^r} = C > 0$ we say a_n converges with rate r.

Examples

- For $a_n = \frac{1}{n}, r = 1$
- For $a_n = \frac{1}{2^{2^n}}, r = 2$

Exercise

Prove the rate of convergence of the interval bisection method is $r = 1$ and $C = 1/2$.

Fixed Point

Definition

A value x^* is called a *fixed point* of the function g if $g(x^*) = x^*$.

Why is it useful?

Sometimes, the root finding problem can be converted to a fixed point finding problem. Example:

Consider the problem of computing the root of $x - 0.2\sin x - 0.5 = 0$. This is equivalent of finding the fixed point of $g(x) = 0.2\sin x + 0.5$.

Contraction Mapping Theorem

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is contractive in $S \subset \mathbb{R}^n$ if there exists $0 < \gamma < 1$ such that for all $x, y \in \mathbb{R}^n$, the following holds: $\|f(x) - f(y)\| \leq \gamma \|x - y\|$.

Theorem

If g is contractive in a closed $S \subset \mathbb{R}^n$ and $g(S) \subset S$ then g has a fixed point in S, i.e. the equation $g(x) = x$ has a solution in S.

Fixed Point Iteration method

Sketch of the iteration method

Let g be a smooth function. Define the sequence $x_{n+1} = g(x_n)$ with x_0 an arbitrary value. If $x^* = g(x^*)$ and $|g'(x^*)| < 1$ then it is not hard to show that $\lim_{n\to\infty} x_n = x^*$.

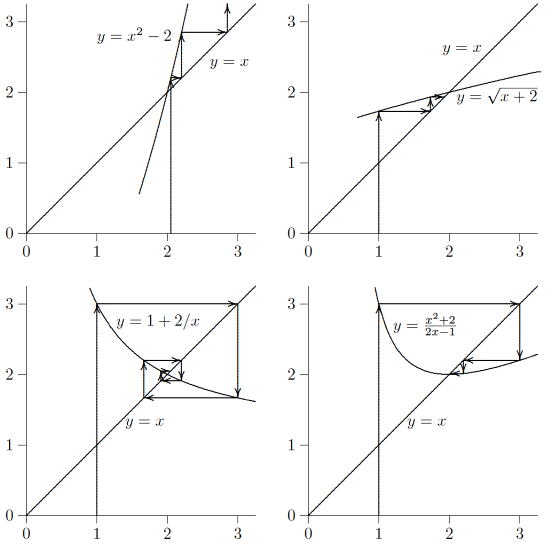
Indeed, the error
$e_{k+1} = x_{k+1} - g(x^*) = g(x_k) - g(x^*) = g'(\theta_k)(x_k - x^*) = g'(\theta_k)e_k$
by Mean Value Theorem. This implies
$|e_{k+1}| \leq C|e_k| \leq \cdots \leq C^k|e_0|$ which converges to 0 since $|C| < 1$.

Fixed Point Example

Write $x^2 - x - 2 = 0$ as a fixed point problem

- $g(x) = x^2 - 2, g'(2) = 4$, so method diverges
- $g(x) = \sqrt{x+2}, g'(2) = 1/4$, so method converges
- $g(x) = 1 + 2/x, g'(2) = -1/2$, so method converges
- $g(x) = (x^2+2)/(2x-1), g'(2) = 0$, so method converges

Fixed Point Solution



References

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