CSC338 Winter 2022

Week 3 - Systems of Linear Equations

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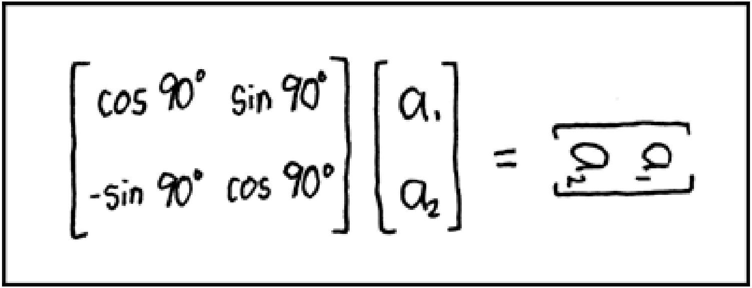
WELCOME TO THE MATRIX



https://xkcd.com/184/

Vector Spaces

What is a vector space over reals?

- A tuple $(V, +, \cdot)$
- V i the set of vectors
- Vector addition, denoted by $+$, and scalar multiplication, denoted by $\cdot$, often omitted, satisfy usual algebraic properties, i.e.:
  - $\forall v, w \in V, \forall \alpha \in \mathbb{R}, v + \alpha w \in V$ (Closure)
  - $\exists 0 \in V, \forall v \in V, v + 0 = v$ (zero vector exists)
  - $\forall v \in V, \exists w \in V, v + w = 0$ (the inverse: $w = -v$)
  - $\forall v \in V, \forall w \in v, v + w = w + v$ (commutativity)
  - $\forall u, v, w \in V, (u + v) + w = u + (w + v)$ (associativity)
  - $\forall \alpha \in \mathbb{R}, \forall v, w, \in V, \alpha(v + w) = \alpha v + \alpha w$
  - $\forall v \in V\ \forall \alpha, \beta \in \mathbb{R}, (\alpha + \beta)v = \alpha v + \beta v$
  - $\forall v \in V, 1 \cdot v = v$

Examples

$\mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

$$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}$$

Functions also form a vector space

Let $f, g : \mathbb{R} \to \mathbb{R}$. Then:

$$f + g : x \mapsto f(x) + g(x)$$

$$\alpha f : x \mapsto : \alpha f(x)$$

are also well defined functions.

More examples

Position of a robotic arm can be represented as vector

Velocity of the ball can be represented as a vector

Linear independence

Linear span and independence

- Let $v_1, \ldots, v_k \in V$, where V is a real vector space.
- Linear span of $v_1, \ldots, v_k$ is called the set $\{a_1 v_1 + \cdots + a_k v_k | a_1, \ldots a_k \in \mathbb{R}\}$.
- A set of vectors $\{v_1, \ldots, v_k\}$ is called linearly independent if

$$\forall a_1, \ldots, a_k \in \mathbb{R}, a_1 v_1 + \cdots + a_k v_k = 0 \implies a_1 = \cdots = a_k = 0.$$

Basis and dimension

Basis of a vector space V. Dimension.

- A set $B = \{v_1, \ldots, v_k\} \subset V$ is called a *basis* if V if:
  - B is linearly independent and spans V
  - Any set of vectors B' such that $B' \supset B$ (in the strict sense), is linearly dependent, i.e. B is maximal.
  - If a vector space B has a finite basis, it can be shown that any other basis of V is going to have same number of vectors. Therefore the cardinality of B is invariant of V, called *dimension*. In this case, V is called finite dimensional space.
  - Otherwise, a vector space is called infnite dimensional.
  - In this course, we will study finite dimensional spaces only.

Linear transformations.

What happens in vectors spaces stays in vector spaces

- Let V, W be two real vector spaces.
- A *linear transformation* is a map $T : V \to W$ satisfying

$$\forall v_1, v_1 \in V, \forall \alpha_1, \alpha_2 \in \mathbb{R}, T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$$

- Example, the linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x\cos\theta + y\sin\theta \\ y\cos\theta - x\sin\theta \end{bmatrix}$$

rotates a vector arm with end point coordinates (x, y) by θ around origin.

MATRICES.

WHAT IS A MATRIX?

By definition, a $n \times m$ real matrix is an ordered set of m vectors from $\mathbb{R}^n$ space. For example, here is a 2×3 real matrix:

$$\begin{bmatrix} 3 & -1 & 4 \\ 0.2 & 9 & -1.2 \end{bmatrix}$$

Matrices.

Where do matrices come from?

Let $V = \mathbb{R}^3, W = \mathbb{R}^2$ be the usual 3-dimensional and 2-dimensional Euclidian spaces. Also let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

be the standard basis for V.

Matrices.

Where do matrices come from?

For any $T : V \to W$, can write due to linearity:

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = xT \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + yT \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + zT \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This means we can write out the image of any vector from $\mathbb{R}^3$ as long as we know how does T transform the vectors of the basis. The images of e_1, e_2, e_3 will be three vectors from $\mathbb{R}^2$. Writing a_{ij} for the $i-$the coordinate of $T(e_j)$ we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

the *matrix representing transformation* T.

MATRICES.

USING ORDINARY RULES OF MATRIX MULTIPLICATION:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

usually called a linear system with two equations and three variables, where

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \end{bmatrix}$$

In this course, we will deal with systems of equations that have equal nymber of equations and variables.

Problems we aim to solve.

Robotic arm

Where should we mound the loose end of the robotic arm so it can reach the point $(0,1)$ if rotated by $\pi/6$?

$$\begin{bmatrix} \cos\pi/6 & \sin\pi/6 \\ -\sin\pi/6 & \cos\pi/6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What is a linear system?

Definitions

Note: in what follows, vectors will be assumed column vectors.

Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

a $n \times n$ real matrix and $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^n$. Sometimes we will denote matrices as follows: $\boldsymbol{A} = [a_{ij}]_{n \times n}$, or, if n is clear from the context, simply $\boldsymbol{A} = [a_{ij}]$.

A *linear system* will be called an equation of the form

$$\boldsymbol{Ax} = \boldsymbol{b}.$$

Matrix operations

Definitions

Let $\boldsymbol{A} = \left[a_{ij}\right], \boldsymbol{B} = \left[b_{ij}\right]$. Then:

$$\boldsymbol{A} + \boldsymbol{B} = \left[a_{ij} + b_{ij}\right]$$

$$\boldsymbol{AB} = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]$$

Identity matrix is called the special matrix $\boldsymbol{I} = \left[\delta_{ij}\right]$ for $\delta_{ij} = 1$ if $i \neq j$ and $\delta_{ii} = 0$. Example $(n = 2)$:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$. Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Inverse of a matrix

Definitions

Inverse of a matrix $\boldsymbol{A}$ (if it exists) is called a matrix denoted by $\boldsymbol{A}^{-1}$ having the following property:

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}.$$

Exercise: Prove that if the inverse of a matrix exists, it is unique.

Recall *determinant* of a matrix is called the number computed by the formula

$$\det(\boldsymbol{A}) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma_i}$$

where S_n is the set of all permutations of numbers $1, 2, \ldots, n$ and the sign of a permutation is the number of its inversions.

Determinant example

Example

Let
$n = 3, S_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$.
Notice $(1,2,3)$ has zero inversions, so its sign is 1, whereas $(1,3,2)$ has one invesion, so its sign is -1 and so on.

$$\det\left(\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\right) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$

$$-a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} = 4$$

Note: Determinant has the following important property: $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.

Invertible (nonsingular) matrices

TFAE

A matrix if invertible (or non-degenerate, or nonsingular) if

- It has an inverse: $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$
- $\det(\boldsymbol{A}) \neq 0$
- $rank(\boldsymbol{A}) = n$ (note: rank of a matrix is the maximum number of linearly independent rows or columns, considered as $\mathbb{R}^n$ vectors.)
- For any nonzero vector $\boldsymbol{x}$, it is true that $\boldsymbol{A}\boldsymbol{x} \neq 0$.

Otherwise the matrix is called singular.

Solving linear systems

$Ax = b$

Let A be nonsingular. Then A^{-1} exists, so

$$x = A^{-1}b$$

shows the solution exists and it is unique.

If A is singular, the system may or may not have solutions. For example

$$\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

has no solution (why?).

Solving linear systems

$\boldsymbol{Ax} = \boldsymbol{b}$

Now let $\boldsymbol{A}$ be singular and assume there is a solution $\boldsymbol{x}_0$ so that $\boldsymbol{Ax}_0 = \boldsymbol{b}$. Since $\boldsymbol{A}$ is singular, there exists a vector $\boldsymbol{y} \neq 0$ such that $\boldsymbol{Ay} = \boldsymbol{0}$ (why?). Then

$$\boldsymbol{A}(\boldsymbol{x}_0 + t\,\boldsymbol{y}) = \boldsymbol{Ax}_0 + t\boldsymbol{Ay} = \boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}$$

so $\boldsymbol{x}_0 + t\boldsymbol{y}$ is another solution for any t.

Summary - solving $\boldsymbol{Ax} = \boldsymbol{b}$

- $\boldsymbol{A}$ nonsingular: unique solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$
- $\boldsymbol{A}$ singular: no solution if $\boldsymbol{b} \notin \operatorname{img} \boldsymbol{Ax}$
- $\boldsymbol{A}$ singular: infinitely many solutions if $\boldsymbol{b} \in \operatorname{img} \boldsymbol{Ax}$

Solving linear systems

$Ax = b$, A upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}, a_{ij} = 0, i > j$$

Assuming $a_{ii} \neq 0$, one can successively compute

$$x_n = \frac{b_n}{a_{nn}}, x_i = \frac{1}{a_{ii}}(b_i - \sum_{k=i+1}^{n} a_{ik}x_k), 1 \leq i \leq n-1$$

Back-Substitution

- *Back-substitution* for upper triangular system $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{b}$

$$x_n = b_n / u_{nn}, \quad x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) / \, u_{ii}, \quad i = n-1, \dots, 1$$

for $j = n$ **to** 1 { loop backwards over columns }
    if $u_{jj} = 0$ **then** stop { stop if matrix is singular }
    $x_j = b_j / u_{jj}$ { compute solution component }
    for $i = 1$ **to** $j - 1$
        $b_i = b_i - u_{ij} x_j$ { update right-hand side }
    end
end

SOLVING LINEAR SYSTEMS

$Ax = b$, A LOWER TRIANGULAR

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, a_{ij} = 0, i < j$$

Assuming $a_{ii} \neq 0$, one can successively compute

$$x_1 = \frac{b_1}{a_{11}}, x_i = \frac{1}{a_{ii}}(b_i - \sum_{k=1}^{i-1} a_{ik} x_k), 2 \leq i \leq n$$

Forward-Substitution

- *Forward-substitution* for lower triangular system $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{b}$

$$x_1 = b_1/\ell_{11}, \quad x_i = \left(b_i - \sum_{j=1}^{i-1} \ell_{ij} x_j \right) / \, \ell_{ii}, \quad i = 2, \ldots, n$$

for $j = 1$ **to** n { loop over columns }
  if $\ell_{jj} = 0$ **then** stop { stop if matrix is singular }
  $x_j = b_j/\ell_{jj}$ { compute solution component }
  for $i = j + 1$ **to** n
    $b_i = b_i - \ell_{ij} x_j$ { update right-hand side }
  end
end

Solving linear systems

How to simplify a linear system?

Let T be a nonsingular matrix. The following computation

$$TAx = Tb \implies x = (TA)^{-1}Tb = A^{-1}T^{-1}Tb = A^{-1}Ib = A^{-1}b$$

shows the solution to a linear ysstem does not change when we multiply both sides with a nonsingular matrix.

Solution Strategy - Gauss Elimination

Multiply both sides of the given system with nonsingular matrices until we obtain an upper triangular system.

Gauss Elimination

A first example

$$\boldsymbol{Ax} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

Want to convert a_{21}, a_{31}, a_{32} to zero. To this end, multiply first row by 2 and subtract it from second row, applying same operation to the right hand side:

$$\boldsymbol{Ax} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

Gauss Elimination

A first example - continued

Multiply first row to -1 and subtract it from third row:

$$\boldsymbol{Ax} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix} = \boldsymbol{b}$$

Subtract second row from third row:

$$\boldsymbol{Ax} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \boldsymbol{b}$$

Gauss Elimination

A first example - back substitution

$$x_3 = 8/4 = 2, x_2 = 4 - x_3 = 2, x_1 = (2 - 4 \cdot 2 + 2 \cdot 2)/2 = -1$$

Question

How to generalize?

Elementary Elimination Matrices

We are interested to transform a column of matrix $\boldsymbol{A}$ as follows:

$$\begin{bmatrix} a_{1k} \\ \dots \\ a_{kk} \\ a_{k+1,k} \\ \dots \\ a_{nk} \end{bmatrix} \rightarrow \begin{bmatrix} a_{1k} \\ \dots \\ a_{kk} \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

Not very hard to see, the following matrix will do (note $\det(\boldsymbol{M_k}) = 1$):

$$\boldsymbol{M_k} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & -\frac{a_{k+1,k}}{a_{kk}} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -\frac{a_{nk}}{a_{kk}} & 0 & \dots & 1 \end{bmatrix}$$

Properties of Elementary Elimination Matrices

- In order to transfor a matrix to upper triangular, using elimination matrices, we need $n-1$ of them: $\boldsymbol{M_1}, \ldots, \boldsymbol{M_{n-1}}$.
- Each $\boldsymbol{M_k}$ is lower triangular and nonsingular
- Recall the transposition operator and let

$$\boldsymbol{m_k} = \begin{bmatrix} 0 & \ldots & 0 & -\frac{a_{k+1,k}}{a_{kk}} & \ldots & -\frac{a_{nk}}{a_{kk}} \end{bmatrix}^T$$

and $\boldsymbol{e_k}$ the $k-$th column of the indentity matrix, it's not hard to see that

$$\boldsymbol{M_k} = \boldsymbol{I} - \boldsymbol{m_k}\boldsymbol{e_k^T}$$

- Check (exercise!)

$$\boldsymbol{M_k^{-1}} = \boldsymbol{I} + \boldsymbol{m_k}\boldsymbol{e_k^T}$$

and denote $\boldsymbol{L_k} = \boldsymbol{M_k^{-1}}$.

Gauss Elimination

- Consider the system $\boldsymbol{Ax} = \boldsymbol{b}$, assuming $a_{ii} \neq 0$ for all i, we may actually apply matrices $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_{n-1}$ to our system and notice the solution does not change, so the following system

$$\boldsymbol{M}_{n-1} \ldots \boldsymbol{M}_1 \boldsymbol{Ax} = \boldsymbol{M}_{n-1} \ldots \boldsymbol{M}_1 \boldsymbol{b}$$

  has same solution as the original system.
- The matrix $\boldsymbol{U} = \boldsymbol{M}_{n-1} \ldots \boldsymbol{M}_1 \boldsymbol{A}$ is upper-triangular, so the system $\boldsymbol{Ux} = \boldsymbol{M}_{n-1} \ldots \boldsymbol{M}_1 \boldsymbol{b}$ can be solved using back-substitution.

LU Factorization

- Recall $\boldsymbol{M}_k^{-1} = \boldsymbol{L}_k$ are lower triangular and product of lower triangular matrices is also lower triangular.
- $\boldsymbol{Ux} = \boldsymbol{M}_{n-1} \ldots \boldsymbol{M}_1 \boldsymbol{b} \implies \boldsymbol{LUx} = \boldsymbol{b}$ letting $\boldsymbol{L} = \boldsymbol{M}_1^{-1} \ldots \boldsymbol{M}_{n-1}^{-1}$.
- As an easy exercise, prove that $\boldsymbol{A} = \boldsymbol{LU}$. This is precisely LU factorization of a nonsingular matrix $\boldsymbol{A}$ with nonzero diagonal elements.

When Gauss Elimination breaks

What happens when $a_{ii} = 0$?

- Gaussian elimination breaks down if leading diagonal entry of remaining unreduced matrix is zero at any stage
- Example:

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \color{red}{0} & 2 \\ 0 & 0 & \color{blue}{2} & 2 \end{bmatrix}$$

- We are unable to cancel the blue colored item due to the zero value (in red).
- Clearly, we need to permute lines 3 and 4.

Permuting the rows

What is a permutation matrix?

- We permute two rows by using a *permutation matrix*.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- The permutation matrix to the left has been obtained from identity matrix permuting rows 3 and 4.
- This implies the general rule to obtain the upper triangular matrix $\boldsymbol{U}$ is:

$$\boldsymbol{M}_{n-1}\boldsymbol{P}_{n-1}\ldots\boldsymbol{M}_1\boldsymbol{P}_1\boldsymbol{A} = \boldsymbol{U}$$

NOT EVERY INVERTIBLE MATRIX AS AN *LU* FACTORIZATION

WHAT IS A PERMUTATION MATRIX?

- Observe the permutation matrices are invertible.
- In general, $P_1^{-1}M_1^{-1}\dots P_{n-1}^{-1}M_{n-1}^{-1}$ is not lower triangular.
- Example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- To make it worse, there are non-invertible matrices that have ***LU*** factorization.
- Example: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Partial pivoting

What is a partial pivoting?

- Let $P = P_{n-1} \ldots P_1$.
- Then $PA = LU$.

More trouble with Gauss Elimination

What happens when $|a_{ii}|$ is small?

- Gaussian elimination may lead to large errors if pivot is very small
- Example:

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0.0001 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

- Canceling the blue item involves dividing with a very small number, leading to error propagation.
- Also this points to poor conditioning.
- In this case, permuting columns 3 and 4 helps.

Complete pivoting

What is a complete pivoting?

- Let Q be the product of permutation matrices used to permute columns
- Then $PAQ = LU$.
- Numerical stability of complete pivoting theoretically superior, but pivot search more expensive than partial pivoting
- Numerical stability of partial pivoting more than adequate in practice, so almost always used in solving linear systems by Gaussian elimination

Gauss Elimination

- Let $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is an $n \times n$ non-singular matrix, and $\boldsymbol{x}, \boldsymbol{b}$ column vectors from $\mathbb{R}^n$.
- By conducting the appropriate row reduction operations, combined with necessary row permuations, we transform our system to $\boldsymbol{M}_{n-1}\boldsymbol{P}_{n-1} \ldots \boldsymbol{M}_1\boldsymbol{P}_1\boldsymbol{Ax} = \boldsymbol{M}_{n-1}\boldsymbol{P}_{n-1} \ldots \boldsymbol{M}_1\boldsymbol{P}_1\boldsymbol{b}$ or $\boldsymbol{Ux} = \boldsymbol{b}'$ where $\boldsymbol{U} = \boldsymbol{M}_{n-1}\boldsymbol{P}_{n-1} \ldots \boldsymbol{M}_1\boldsymbol{P}_1$ is upper-triangular and $\boldsymbol{b}' = \boldsymbol{M}_{n-1}\boldsymbol{P}_{n-1} \ldots \boldsymbol{M}_1\boldsymbol{P}_1\boldsymbol{b}$.
- This transformation is know as **Gauss Elimination**. By checking out our implementation (completed in tutorial and homework), we conclude its computational complexity is of the order $\mathcal{O}(n^3)$.
- To complete the solution of our system, we just need to perform the backward substitution, whose complexity is $\mathcal{O}(n^2)$.
- Therefore, the allover complexity is $\mathcal{O}(n^3)$.

Gauss Elimination Example

Example

$$\boldsymbol{Ax} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

After a few suitable row reductions, we have completed Gauss Elimination:

$$\boldsymbol{Ux} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \boldsymbol{b}'$$

Every solution of the original system is a solution of the transformed system and vice versa. Therefore, we obtain the solution to the original system by performing backward substitution to the transformed system.

Gauss Jordan Elimination Example

Example

By continuing elimination of the elements in the upper triangle of the matrix:

$$\boldsymbol{Ux} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \boldsymbol{b}'$$

we obtain:

$$\boldsymbol{Ux} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 8 \end{bmatrix} = \boldsymbol{b}'$$

After diagonalization, the solution can be easily computed by divided RHS components by diagonal elements.

Gauss Jordan Elimination

The algorithm

- Transform the system in upper triangular form
- Eliminate elements above the main diagonal
- The **Gauss Jordan Elimination** process finishes by producing a diagonal matrix.

Solving the system and complexity

After diagonalization, the solution can be easily computed by divided RHS components by diagonal elements. The complexity of the last operation is $O(n)$, however the elimination of the upper triangular elements is of the order $O(n^3)$, so we do not gain in overall complexity.

References

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