CSC338 Exam Grading Guide

Question 1. Machine Precision [4 pt]

In most floating-point systems, a quick approximation of the machine precision ϵ_{mach} can be obtained by evaluating the expression $\epsilon_{\text{mach}} \approx$ `|3 * (4 / 3 - 1) - 1|`.

We'll work with the floating point system $F(\beta = 10, p = 6, L = -100, U = 100)$. You can assume that chopping is used for rounding.

Part a. [3 pt]

Perform the floating-point computation `|3 * (4 / 3 - 1) - 1|` in this floating-point system. Show the partial result of each step of the floating-point computation in the order that the computations occur.

Solutions:

1. The division `4.00000 * 10^0` / `3.00000 * 10^0` should yield `1.33333 * 10^0`.
2. The subtraction `1.00000 * 10^0 - 1.33333 * 10^0` should yield `3.33330 * 10^(-1)`. (**NOT** `0.33333 * 10^0` or `0.33333`)
3. The multiplication `3.00000 * 10^0` × `3.33330 * 10^(-1)` should yield `9.99990 * 10^(-1)`. (**NOT** `0.99999 * 10^0` or `0.99999`)
4. The subtraction `9.99990 * 10^(-1)` - `1.00000 * 10^0` should yield `-1.00000 * 10^(-5)` (see below)
5. Applying the absolute value yields `1.00000 * 10^(-5)`

```
1.0 0000    10^0
  9.99990   10^(-1)
----------
       1.0
```

Grading:

- +1 point for the right answer
- +0.5 point for the applying the absolute value
- +0.5 point for using the correct precision at each step
- +1 point for showing the floating-point representation in each step (with half point for missing one of the results)

Part b. [1 pt]

Solutions: Should be $\beta^{1-p} = 10^{-5}$.

Grading: No part marks.

Q2. Condition Numbers [4 pt]

Consider the function $f(x) = x^4 + x^2 - x - 1$. Compute the following condition numbers, accurate to 3 significant decimal digits:

Part a. [1 pt]

The relative condition number of evaluating the function $f(x)$. Your answer should be a function of x.

Solution: $\left|\frac{(4x^4+2x^2-x)}{x^4+x^2-x-1}\right|$

Grading: Part mark for missing absolute value

Part b. [1 pt]

The absolute condition number of finding the root of $f(x)$ at $x = 1$.

Solution: $\left|\frac{1}{4x^3+2x-1}\right| = \frac{1}{5}$

Grading: Part mark for not actually compint the absolute value at $x = 1$.

Part c. [2 pt]

Consider the problem of finding the minima of $f(x)$ at $x = 0.38546$. We mentioned in class that this problem is not well conditioned, but that there is another problem with the same solution. What is this other problem that we could solve instead? What is the absolute condition number of that problem?

Solution: The problem is finding a root of $f'(x)$ at the same x. The conditioning of this problem is $\frac{1}{f''(x)} = \frac{1}{12x^2+2} = 0.1509$

Grading:

- +1 point for identifying the correct problem
- +0.5 point for the correct CN expression
- +0.5 point for evaluating CN expression at the root.

Q3. QR Factorization [4 pt]

Consider the overdetermined $m \times n$ system $A\mathbf{x} = \mathbf{b}$.

Show that if the $m \times m$ matrix Q is orthogonal, then multiplying both sides of the equation by Q will preserve the 2-norm of the residual $||A\mathbf{x} - \mathbf{b}||_2$.

Solutions: If we apply Q to both side of the equation $A\mathbf{x} = \mathbf{b}$, the new equation becomes $QA\mathbf{x} = Q\mathbf{b}$. We wish to show that $||A\mathbf{x} - \mathbf{b}||_2 = ||QA\mathbf{x} - Q\mathbf{b}||_2$, or, equivalently that $||A\mathbf{x} - \mathbf{b}||_2^2 = ||QA\mathbf{x} - Q\mathbf{b}||_2^2$

Let $\mathbf{r} = A\mathbf{x} - \mathbf{b}$, then

$$\begin{aligned}
||QA\mathbf{x} - Q\mathbf{b}||_2^2 &= ||Qr||_2^2 \\
&= (Qr)^T(Qr) \\
&= r^TQ^TQr \\
&= r^TIr \\
&= r^TIr \\
&= ||r||_2^2 \\
&= ||A\mathbf{x} - \mathbf{b}||_2^2
\end{aligned}$$

Grading:

- +1 point for identifying what we need to prove
- +3 point for the rest of the proof, subtracting 1-2 points per error

Q4. Cholesky Factorization [3 pt]

Perform Cholesky Factorization on this matrix. Show all your steps.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

Solutions

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

Grading: 0.5 pt for computing each of the six elements, where work is shown

Q5. Householder Transforms [6 pt]

Suppose you are using Householder transformations to compute the QR factorization of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 5 & 5 \\ 2 & 3 & 9 \\ 3 & 1 & 1 \\ 2 & 5 & 1 \end{bmatrix}$$

Part a. [1 pt]

How many Householder transformations are required?

Solutions: 3

Grading: No part marks

Part b. [2 pt]

Specify the first Householder transformation by finding the vector v describing the transformation.

Solutions: We would like to zero out below the diagonal in the first column of A, whose norm is $\sqrt{4+4+4+9+4} = 5$. So

$$\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

Grading:

- +1 point for computing the norm of the vector
- +1 point for the correct computation of v, including choosing the right sigh
- -0.5 point for other typos

Part c. [3 pt]

Apply the first Householder transformation from Part (b) to the matrix A. Draw a box around your final result.

Solutions: Let $\mathbf{a_k}$ be the k-th column of A. Then:

We know that $H\mathbf{a_1} = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, where $\mathbf{a_1}$ is the first column of A. For the other two columns of A we need to do some work:

$$
\begin{aligned}
H\mathbf{a_1} &= \mathbf{a_1} - 2\mathbf{v}\frac{\mathbf{v}^T\mathbf{a_1}}{\mathbf{v}^T\mathbf{v}} \\
&= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 7 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix}\frac{35}{70} \\
&= \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
H\mathbf{a_2} &= \mathbf{a_2} - 2\mathbf{v}\frac{\mathbf{v}^T\mathbf{a_2}}{\mathbf{v}^T\mathbf{v}} \\
&= \begin{bmatrix} 3 \\ 5 \\ 3 \\ 1 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 7 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix}\frac{50}{70} \\
&= \begin{bmatrix} -7 \\ 15/7 \\ 1/7 \\ -23/7 \\ 15/7 \end{bmatrix} \\
&= \begin{bmatrix} -7 \\ 2.14 \\ 0.14 \\ -3.29 \\ 2.14 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
H\mathbf{a_3} &= \mathbf{a_3} - 2\mathbf{v}\frac{\mathbf{v}^T\mathbf{a_3}}{\mathbf{v}^T\mathbf{v}} \\
&= \begin{bmatrix} 3 \\ 5 \\ 3 \\ 1 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 7 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix}\frac{50}{70} \\
&= \begin{bmatrix} -7 \\ 19/7 \\ 47/7 \\ -17/7 \\ -9/7 \end{bmatrix} \\
&= \begin{bmatrix} -7 \\ 2.71 \\ 6.71 \\ -2.43 \\ -1.29 \end{bmatrix}
\end{aligned}
$$

Grading:

- 1 point per column
- part marks for minor issues **only if student shows their work**
- -0.5 pt for drawing a box around only part of the matrix as the solution

Q6. Root-Finding Algorithms [4 pt]

We are running a root-finding algorithm to find the root of a function f, and obtain the following output showing the estimate of the root in each iteration.

```
Iteration: 0     x = 3.0000000000000000
Iteration: 1     x = 2.1530576920133857
Iteration: 2     x = 1.9540386420058038
Iteration: 3     x = 1.9339715327520701
Iteration: 4     x = 1.9337537885576270
Iteration: 5     x = 1.9337537628270216
Iteration: 6     x = 1.9337537628270212
Iteration: 7     x = 1.9337537628270212
Iteration: 8     x = 1.9337537628270212
Iteration: 9     x = 1.9337537628270212
Iteration: 10    x = 1.9337537628270212
```

Part a [2 pt]

Is the convergence rate best described as linear, superlinear (but not quadratic), or quadratic? Explain your choice.

Solution: Quadratic. The difference between iterations decrease quadratically. The correct number of digits roughly double in in each iteration.

Grading: One point for the answer. One point for a reasonable explanation that looks at either number of correct digits, or the change in x in each iteration.

Part b [2 pt]

What root-finding algorithm do you think is applied? Explain your choice.

Solution: Either Newton's Method or Fixed-Point Iteration. These are both algorithms that can have quadratic convergence.

Grading: One point for the answer. One point for a reasonable explanation that considers the convergence rate.

Q7. Fixed-Point Iteration [4 pt]

We wish to find a root of the function $f(x) = x^5 + x^2 - x + 2$ using fixed-point iteration. We know that f has a root near $x = -1.4$.

Part a [2 pt]

Suppose we apply fixed-point iteration on the function $g_1(x) = x^5 + x^2 + 2$. If we start near $x = -1.4$, will the algorithm converge to the desired root?

Solution. No, because $|g_1'(x)|$ near the root is larger than 1.

Grading: Half point for saying "no", and the remaining points for computing the derivative of g_1'.

Part b [2 pt]

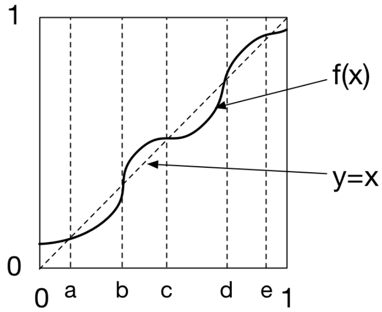
Come up with another function g_2 whose fixed-points are roots of $f(x)$. Show that fixed-point iteration *does* converge for your choice of g_2 if we start near the desired root around $x = -1.4$.

Solution: The function $g_2(x) = (-x^2 + x - 2)^{\frac{1}{5}}$ should do, and has $|g_2'(-1.4)| \approx 0.2$

Grading: One point for a reasonabel g_2 and one point for showing that $|g_2'(x)| < 1$ near the root.

Q8. Fixed-Point Iteration [2 pt]

Consider the function $f(x)$ below:



Suppose we were to use fixed-point iteration to find a fixed point of f. We start at a point x just to the left of d (i.e. just a little less than d). Which fixed point would the fixed-point iteration converge to? Explain your reasoning.

Solutions Should converge to c because $f'(d) > 1$ and $f'(c) < 1$.

Grading +1 point for saying converge to c, +0.5pt for saying that $f'(d) > 1$, +0.5pt for saying $f'(c) > 1$.

Q9. Golden Section Search [4 pt]

We would like to find a local minimum of the function $f(x) = x^4 - x$.

Part a [2 pt]

The function f is unimodal in the interval `[a, b]`, with $a = 0$, $b = 1$. Perform one iteration of Golden Section search, showing all your work. What is your new interval `[a, b]`?

Solutions

$$\begin{aligned} x_1 &= a + (b-a) * 0.382 \\ &= 0.382 \\ x_2 &= a + (b-a) * 0.618 \\ &= 0.618 \\ f(x_1) &= -0.361 \\ f(x_2) &= -0.472 \end{aligned}$$

Since $f(x_2) < f(x_1)$, the minima will not be in the interval $[0, 0.382]$, so the new interval is $a = 0.382$, $b = 1$.

Grading

- +1 pt for the correct x_1 and x_2
- +1 pt for drawing the correct conclusion about the next interval to search
- Points can only be earned if the computations are shown

Part b [2 pt]

Perform another iteration of Golden Section search, starting from your interval in Part (a), showing all your work. What is your new interval `[a, b]`?

Solutions

$$
\begin{aligned}
x_1 &= 0.618 \\
f(x_1) &= -0.472 \\
x_2 &= a + (b - a) * 0.618 \\
&= 0.764 \\
f(x_2) &= -0.423
\end{aligned}
$$

Since $f(x_1) < f(x_2)$, so the new interval is $a = 0.382$, $b = 0.764$.

Grading

- +1 pt for the correct x_1 and x_2
- +1 pt for drawing the correct conclusion about the next interval to search
- Points can only be earned if the computations are shown

Q10. Optimization [7 pt]

Consider the problem of finding a local minimum of the function $f(x_1, x_2) = \sin(x_1)x_2 + \cos(x_2)x_1$. We will start at the estimate $\mathbf{x} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$. (Note: Recall that $\sin(\frac{\pi}{2}) = 1$ and $\cos(\frac{\pi}{2}) = 0$)

Part a. [2 pt]

Compute the gradient $\nabla f(\mathbf{x})$ at the point $\begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$. Show your work.

Solutions

$$
\begin{aligned}
\nabla f(\mathbf{x}) &= \begin{bmatrix} \cos(x_1)x_2 + \cos(x_2) \\ \sin(x_1) - \sin(x_2)x_1 \end{bmatrix} \\
\nabla f(\begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}) &= \begin{bmatrix} 0 \\ 1 - \frac{\pi}{2} \end{bmatrix}
\end{aligned}
$$

Grading One point per computation.

Part b. [2 pt]

Compute the Hessian $H_f(\mathbf{x})$ at the point $\begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$. Show your work.

Solutions

$$
\begin{aligned}
H_f(\mathbf{x}) &= \begin{bmatrix} \sin(x_1)x_2 & \cos(x_1) - \sin(x_2) \\ \cos(x_1) - \sin(x_2) & -cos(x_2)x_1 \end{bmatrix} \\
H_f(\begin{bmatrix} -\frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}) &= \begin{bmatrix} \frac{\pi}{2} & -1 \\ -1 & 0 \end{bmatrix}
\end{aligned}
$$

Grading Half point per computation.

Part c. [2 pt]

We wish to use Newton's Method to find a local minimum of f. The Newton's Method update rule has the form $\mathbf{x} \leftarrow \mathbf{x} - \mathbf{s}$ for some vector $\mathbf{s}$. Write down the system of equations do we need to solve in order to compute $\mathbf{s}$. (You don't need to actually solve the system!)

Solutions We need $H_f\mathbf{s} = \nabla f(\mathbf{x})$, so the system is

$$\begin{bmatrix} \frac{\pi}{2} & -1 \\ -1 & 0 \end{bmatrix} \mathbf{s} = \begin{bmatrix} 0 \\ 1 - \frac{\pi}{2} \end{bmatrix}$$

Grading: Subtract half point for the sign of $\mathbf{s}$ Subtract half point for the sign of $\mathbf{s}$ (the update rule here already has a negative sign)

Part d. [1 pt]

We wish to use gradient descent to find a local minimum of f. Write down the gradient descent update rule starting at the point $\mathbf{x} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$, assuming a learning rate lf $\alpha = 1$.

You don't need to actually compute the update, but should write down all the values necessary to make the computation. (In other words, other than π, sin, and cos, your update rule should not contain any other letters.)

Solutions

$$\begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 - \frac{\pi}{2} \end{bmatrix}$$

Grading: Subtract half point if the sign is wrong.

Q11. Newton's Method and Linear Equations [4 pt]

Recall that when using Newton's Method to optimize $f : \mathbb{R} \to \mathbb{R}$, in each iteration we need to solve systems of equation of the form $H_f(\mathbf{x})\mathbf{s} = -\nabla f(\mathbf{x})$.

Suppose that the fucntion we wish to minimize has a Hessian $H_f(\mathbf{x})$ can be decomposed into a sum of two parts:

$$H_f(\mathbf{x}) = H + \begin{bmatrix} x_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where H is constant and does not depend on $\mathbf{x}$, and the second matrix has a single non-zero entry that depends only on the first element of $\mathbf{x}$.

Explain what strategy you would use when solving these systems of equations, so that we minimize the amount of computations necessary. You can write either pseudocode, or a clear enough description/explanation that a programmer can translate into pseudocode.

Solutions:

The second matrix is clearly rank 1, so that the matrix $H_f(\mathbf{x})$ can be written as $H_f(\mathbf{x}) = H + \mathbf{u}\mathbf{v}^T$, with $\mathbf{u} = \begin{bmatrix} -1 & 0 & \cdots \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_1 & 0 & \cdots \end{bmatrix}$. We can use the Sherman-Morrison formula to solve systems of the form $(H - \mathbf{u}\mathbf{v}^T)\mathbf{s} = \mathbf{b}$ by using the rule:

$$
\begin{aligned}
(H - \mathbf{u}\mathbf{v}^T)^{-1} &= H^{-1} + H^{-1}\mathbf{u}(1 - \mathbf{v}^T H^{-1}\mathbf{u})^{-1}\mathbf{v}^T H^{-1} \\
(H - \mathbf{u}\mathbf{v}^T)^{-1}\mathbf{b} &= H^{-1}\mathbf{b} + H^{-1}\mathbf{u}(1 - \mathbf{v}^T H^{-1}\mathbf{u})^{-1}\mathbf{v}^T H^{-1}\mathbf{b}
\end{aligned}
$$

So the algorithm is as follows:

1. Compute the LU factorization of $H = LU$, and save the matrices L and U.
2. Solve $\mathbf{z} = H^{-1}\mathbf{u}$, and save the solution.
3. *In each iteration*, solve $\mathbf{y} = H^{-1}\mathbf{b}$, then compute $\mathbf{s} = \mathbf{y} + \mathbf{z}\frac{\mathbf{v}^T\mathbf{y}}{1-\mathbf{v}^T\mathbf{z}}$

Grading:

- +1 point for recognizing that the second matrix is rank 1
- +0.5 point for talking about the Sherman-Morrison Formula
- +0.5 point for demonstrating the choice of u, v
- +1 point for clearly showing that we should only compute the LU factorization of H once
- +1 point for showing how to apply the Sherman Morrison formula (choice of u, v, what to compute in each iteration)

Q11. Newton's Method and Linear Equations [4 pt]

Recall that when using Newton's Method to optimize $f : \mathbb{R} \to \mathbb{R}$, in each iteration we need to solve systems of equation of the form $H_f(\mathbf{x})\mathbf{s} = -\nabla f(\mathbf{x})$.

Suppose that the fucntion we wish to minimize has a Hessian $H_f(\mathbf{x})$ can be decomposed into a sum of two parts:

$$
H_f(\mathbf{x}) = H + \begin{bmatrix} x_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}
$$

where H is constant and does not depend on $\mathbf{x}$, and the second matrix has a single non-zero entry that depends only on the first element of $\mathbf{x}$.

Explain what strategy you would use when solving these systems of equations, so that we minimize the amount of computations necessary. You can write either pseudocode, or a clear enough description/explanation that a programmer can translate into pseudocode.

Solutions:

The second matrix is clearly rank 1, so that the matrix $H_f(\mathbf{x})$ can be written as $H_f(\mathbf{x}) = H + \mathbf{u}\mathbf{v}^T$, with $\mathbf{u} = \begin{bmatrix} -1 & 0 & \cdots \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_1 & 0 & \cdots \end{bmatrix}$. We can use the Sherman-Morrison formula to solve systems of the form $(H - \mathbf{u}\mathbf{v}^T)\mathbf{s} = \mathbf{b}$ by using the rule:

$$
\begin{aligned}
(H - \mathbf{u}\mathbf{v}^T)^{-1} &= H^{-1} + H^{-1}\mathbf{u}(1 - \mathbf{v}^T H^{-1}\mathbf{u})^{-1}\mathbf{v}^T H^{-1} \\
(H - \mathbf{u}\mathbf{v}^T)^{-1}\mathbf{b} &= H^{-1}\mathbf{b} + H^{-1}\mathbf{u}(1 - \mathbf{v}^T H^{-1}\mathbf{u})^{-1}\mathbf{v}^T H^{-1}\mathbf{b}
\end{aligned}
$$

So the algorithm is as follows:

1. Compute the LU factorization of $H = LU$, and save the matrices L and U.
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3. *In each iteration*, solve $\mathbf{y} = H^{-1}\mathbf{b}$, then compute $\mathbf{s} = \mathbf{y} + \mathbf{z}\frac{\mathbf{v}^T\mathbf{y}}{1-\mathbf{v}^T\mathbf{z}}$

Grading:

- +1 point for recognizing that the second matrix is rank 1
- +0.5 point for talking about the Sherman-Morrison Formula
- +0.5 point for demonstrating the choice of u, v
- +1 point for clearly showing that we should only compute the LU factorization of H once
- +1 point for showing how to apply the Sherman Morrison formula (choice of u, v, what to compute in each iteration)