CSC338 Final Examination

This exam consists of 11 questions and a total of 46 points. An additional 4 points will be awarded for uploading your aid sheet, and following the rest of the instructions in the exam link PDF file (e.g. uploading only pdf/text/image files, completing your declaration, making sure your images are rotated correctly, etc).

The questions are not ordered in terms of difficulty. If you are writing your solutions on paper, please make sure that your answers are legible. We can't give you marks if we can't understand your solutions. **Good luck!**

Question 1. Machine Precision [4 pt]

In most floating-point systems, a quick approximation of the machine precision ϵ_{mach} can be obtained by evaluating the expression $\epsilon_{\text{mach}} \approx$ `|3 * (4 / 3 - 1) - 1|`.

We'll work with the floating point system $F(\beta = 10, p = 6, L = -100, U = 100)$. You can assume that chopping is used for rounding.

Part a. [3 pt] Perform the floating-point computation `|3 * (4 / 3 - 1) - 1|` in this floating-point system. Show the partial result of each step of the floating-point computation in the order that the computations occur.

Part b. [1 pt] What is ϵ_{mach} for this floating-point system?

Question 2. Condition Numbers [4 pt]

Consider the function $f(x) = x^4 + x^2 - x - 1$. Compute the following condition numbers, accurate to 3 significant decimal digits:

Part a. [1 pt] The relative condition number of evaluating the function $f(x)$. Your answer should be a function of x.

Part b. [1 pt] The absolute condition number of finding the root of $f(x)$ at $x = 1$.

Part c. [2 pt] Consider the problem of finding the minima of $f(x)$ at $x = 0.38546$. We mentioned in class that this problem is not well conditioned, but that there is another problem with the same solution. What is this other problem that we could solve instead? What is the absolute condition number of that problem?

Question 3. QR Factorization [4 pt]

Consider the overdetermined $m \times n$ system $A\mathbf{x} = \mathbf{b}$.

Show that if the $m \times m$ matrix Q is orthogonal, then multiplying both sides of the equation by Q will preserve the 2-norm of the residual $||A\mathbf{x} - \mathbf{b}||_2$.

Question 4. Cholesky Factorization [3 pt]

Perform Cholesky Factorization on this matrix. Show all your steps.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

Question 5. Householder Transforms [6 pt]

Suppose you are using Householder transformations to compute the QR factorization of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 5 & 5 \\ 2 & 3 & 9 \\ 3 & 1 & 1 \\ 2 & 5 & 1 \end{bmatrix}$$

Part a [1 pt] How many Householder transformations are required?

Part b [2 pt] Specify the first Householder transformation by finding the vector v describing the transformation.

Part c [3 pt] Apply the first Householder transformation from Part (b) to the matrix A. Draw a box around your final result.

Question 6. Root-Finding Algorithms [4 pt]

We are running a root-finding algorithm to find the root of a function f, and obtain the following output showing the estimate of the root in each iteration.

```
Iteration: 0     x = 3.0000000000000000
Iteration: 1     x = 2.1530576920133857
Iteration: 2     x = 1.9540386420058038
Iteration: 3     x = 1.9339715327520701
Iteration: 4     x = 1.9337537885576270
Iteration: 5     x = 1.9337537628270216
Iteration: 6     x = 1.9337537628270212
Iteration: 7     x = 1.9337537628270212
Iteration: 8     x = 1.9337537628270212
Iteration: 9     x = 1.9337537628270212
Iteration: 10    x = 1.9337537628270212
```

Part a [2 pt]. Is the convergence rate best described as linear, superlinear (but not quadratic), or quadratic? Explain your choice.

Part b [2 pt]. What root-finding algorithm do you think is applied? Explain your choice.

Question 7. Fixed-Point Iteration [4 pt]

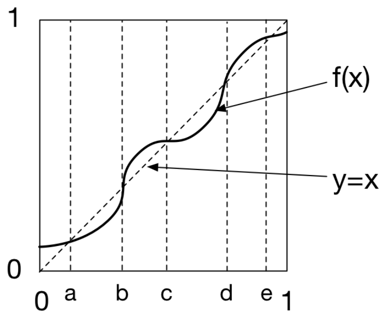
We wish to find a root of the function $f(x) = x^5 + x^2 - x + 2$ using fixed-point iteration. We know that f has a root near $x = -1.4$.

Part a [2 pt] Suppose we apply fixed-point iteration on the function $g_1(x) = x^5 + x^2 + 2$. If we start near $x = -1.4$, will the algorithm converge to the desired root?

Part b [2 pt] Come up with another function g_2 whose fixed-points are roots of $f(x)$. Show that fixed-point iteration *does* converge for your choice of g_2 if we start near the desired root around $x = -1.4$.

Question 8. Fixed-Point Iteration [2 pt]

Consider the function $f(x)$ below:



Suppose we were to use fixed-point iteration to find a fixed point of f. We start at a point x just to the left of d (i.e. just a little less than d). Which fixed point would the fixed-point iteration converge to? Explain your reasoning.

Question 9. Golden Section Search [4 pt]

We would like to find a local minimum of the function $f(x) = x^4 - 2x$.

Part a. [2 pt] The function f is unimodal in the interval $a = 0$, $b = 1$. Perform one iteration of Golden Section search, showing all your work. What is your new interval `[a, b]`?

Part b. [2 pt] Perform another iteration of Golden Section search, starting from your interval in Part (a), showing all your work. What is your new interval `[a, b]`?

Question 10. Optimization [7 pt]

Consider the problem of finding a local minimum of the function $f(x_1, x_2) = \sin(x_1)x_2 + \cos(x_2)x_1$. We will start at the estimate $\mathbf{x} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$. (Note: Recall that $\sin(\frac{\pi}{2}) = 1$ and $\cos(\frac{\pi}{2}) = 0$)

Part a. [2 pt] Compute the gradient $\nabla f(\mathbf{x})$ at the point $\begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$. Show your work.

Part b. [2 pt] Compute the Hessian $H_f(\mathbf{x})$ at the point $\begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$. Show your work.

Part c. [2 pt] We wish to use Newton's Method to find a local minimum of f. The Newton's Method update rule has the form $\mathbf{x} \leftarrow \mathbf{x} - \mathbf{s}$ for some vector $\mathbf{s}$. Write down the system of equations do we need to solve in order to compute $\mathbf{s}$. (You don't need to actually solve the system!)

Part d. [1 pt] We wish to use gradient descent to find a local minimum of f. Write down the gradient descent update rule starting at the point $\mathbf{x} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$, assuming a learning rate lf $\alpha = 1$.

You don't need to actually compute the update, but should write down all the values necessary to make the computation. (In other words, other than π, sin, and cos, your update rule should not contain any other letters.)

Question 11. Newton's Method and Linear Equations [4 pt]

Recall that when using Newton's Method to optimize $f : \mathbb{R} \to \mathbb{R}$, in each iteration we need to solve systems of equation of the form $H_f(\mathbf{x})\mathbf{s} = -\nabla f(\mathbf{x})$.

Suppose that the fucntion we wish to minimize has a Hessian $H_f(\mathbf{x})$ can be decomposed into a sum of two parts:

$$H_f(\mathbf{x}) = H + \begin{bmatrix} x_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where H is constant and does not depend on $\mathbf{x}$, and the second matrix has a single non-zero entry that depends only on the first element of $\mathbf{x}$.

Explain what strategy you would use when solving these systems of equations, so that we minimize the amount of computations necessary. You can write either pseudocode, or a clear enough description/explanation that a programmer can translate into pseudocode.

END OF EXAM