The Moving Average Models MA(1) and MA(2)

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First-order moving-average models

A first-order moving-average process, written as MA(1), has the general equation

$$x_t = w_t + bw_{t-1}$$

where w_t is a white-noise series distributed with constant variance σ_w^2.

The Autocovariance for MA(1) Models

We must compute $\gamma(k)$, which is defined as the autocovariance of the process at lag k. For simplicity, assume that the mean has been subtracted from our data, so that x_t has zero mean. Then

$$\gamma(k) = E(x_t x_{t-k})$$

The Autocovariance for MA(1) Models

$$
\begin{aligned}
\gamma(k) &= E[(w_t + bw_{t-1})(w_{t-k} + bw_{t-k-1})] \\
&= E(w_t w_{t-k} + bw_t w_{t-k-1} + bw_{t-1} w_{t-k} + b^2 w_{t-1} w_{t-k-1}) \\
&= E(w_t w_{t-k}) + E(bw_t w_{t-k-1}) + E(bw_{t-1} w_{t-k}) + E(b^2 w_{t-1} w_{t-k-1})
\end{aligned}
$$

The Autocovariance for MA(1) Models

Now set $k = 0$ and recall that $\gamma(0) = \sigma^2_{MA}$, the variance of your series.

$$\gamma(0) = \sigma^2_{MA} = E(w_t^2) + bE(w_t w_{t-1}) + bE(w_{t-1} w_t) + b^2 E(w_{t-1}^2)$$

$$\gamma(0) = \sigma^2_{MA} = \sigma^2_w + 0 + 0 + b^2\sigma^2_w = (1 + b^2)\sigma^2_w.$$

The Autocovariance for MA(1) Models

Now set $k = 1$.

$$\gamma(1) = E(w_t w_{t-1}) + bE(w_t w_{t-2}) + bE(w_{t-1}^2 w_{t-1}) + b^2 E(w_{t-1} w_{t-2})$$

$$\gamma(1) = b\sigma_w^2.$$

The Autocovariance for MA(1) Models

For $k > 1$, we will obtain $\gamma(k) = 0$, since $E[(w_t + bw_{t-1})(w_{t-k} + bw_{t-k-1})]$ will contain only terms whose expected value is zero.

Note. For an $MA(1)$, the autocovariance function truncates (i.e., it is zero) after lag 1.

The Autocorrelation for MA(1) Models

$$\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1.$$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{b}{1+b^2}.$$

$$\rho(k) = 0 \quad \text{for all } k > 1.$$

The Autocovariance for MA(q) Models

For the qth-order MA process, we can use a similar derivation to show that the autocovariance function, $\gamma(k)$, truncates after lag q. Once again

$$\gamma(k) = E(x_t x_{t-k})$$

The Autocovariance for MA(q) Models

For $k = 0$, we obtain

$$\gamma(0) = \sigma^2_{MA} = (b_0^2 + b_1^2 + b_2^2 + ... + b_q^2)\sigma_w^2.$$

For $k = 1$, we obtain

$$\gamma(1) = (b_1 b_0 + b_2 b_1 + ... + b_q b_{q-1})\sigma_w^2.$$

The Autocovariance for MA(q) Models

In general, we obtain the basic equation

$$\gamma(k) = \sigma_w^2 \sum_{s=0}^{q} b_s b_{s-k}.$$

Second-order Moving-Average Models

Consider the MA(2) process, which is given by

$$x_t = w_t + b_1 w_{t-1} + b_2 w_{t-2},$$

where w_t is again a white-noise process.

MA(2), Autocovariance function

At this point, it should be easy to see that

$\gamma(0) = \sigma^2_{MA} = (1 + b_1^2 + b_2^2)\sigma^2_w$

$\gamma(1) = (b_1 + b_1 b_2)\sigma^2_w$

$\gamma(2) = b_2\sigma^2_w$

$\gamma(k) = 0$ for $k > 2$.

MA(2), Autocorrelation function

$\rho(0) = 1$
$\rho(1) = \frac{b_1 + b_1 b_2}{1 + b_1^2 + b_2^2}$
$\rho(2) = \frac{b_2}{1 + b_1^2 + b_2^2}$
$\rho(k) = 0$ for $k > 2$.
Thus, we see that the autocorrelation function for an MA(2) process truncates after two lags.

MA(1) is an AR(∞)

Suppose that we have an MA(1) model

$$x_t = w_t + bw_{t-1}.$$

Then,

$$x_{t-1} = w_{t-1} + bw_{t-2}.$$

Solve this equation for w_{t-1} and substitute the result back into $x_t = w_t + bw_{t-1}$.

MA(1) is an AR(∞)

This gives

$$\begin{aligned} x_t &= w_t + b(x_{t-1} - bw_{t-2}) \\ &= bx_{t-1} + w_t - b^2 w_{t-2} \end{aligned}$$

(Now, we repeat the process with w_{t-2})

MA(1) is an AR(∞)

$$x_{t-2} = w_{t-2} + bw_{t-3}.$$

Solve this equation for w_{t-2} and substitute the result back into $x_t = bx_{t-1} + w_t - b^2 w_{t-2}$.

$$x_t = bx_{t-1} - b^2 x_{t-2} + w_t + b^3 w_{t-3}$$

MA(1) is an AR(∞)

We can continue indefinitely as long as b^s goes to zero (i. e., $|b| < 1$) to obtain

$$x_t = w_t + bx_{t-1} - b^2 x_{t-2} + b^3 x_{t-3} - ... + ...$$

This is an AR(∞) process, but it only holds under the **invertibility condition** that $|b| < 1$.

More about invertibility

Consider the following first-order MA processes:

A: $x_t = w_t + \theta w_{t-1}$

B: $x_t = w_t + \frac{1}{\theta} w_{t-1}$

More about invertibility

It can easily be shown that these two different processes have exactly the same autocorrelation function (Right?)

$$\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1.$$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{1+\theta^2}.$$

$$\rho(k) = 0 \quad \text{for all } k > 1.$$

More about invertibility

If $|\theta| < 1$, the series $(AR(\infty))$ for A converges whereas that for B does not. Thus if $|\theta| < 1$, model A is said to be invertible whereas model B is not. **The imposition of the invertibility condition ensures that there is a unique MA process for a given autocorrelation function**.

Simulated Examples of the MA(1) Model

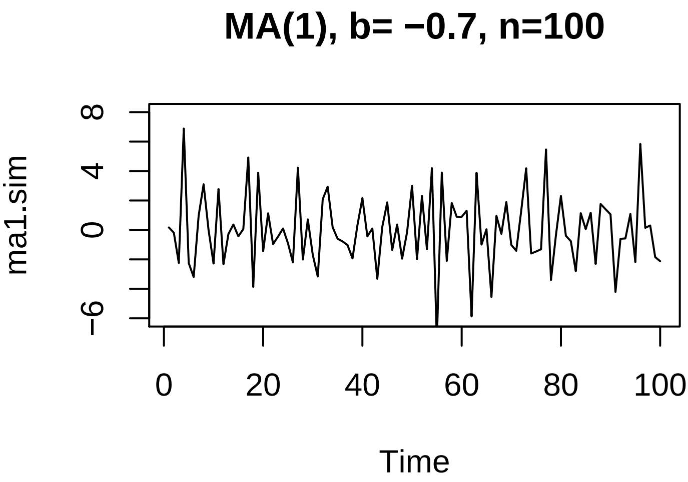
$$x_t = w_t + b_1 w_{t-1}$$

There are two cases, positive and negative values.
Case i) $b_1 = -0.7$
Case ii) $b_1 = 0.3$.

R Code

```
set.seed(9999);

# simulating MA(1);
ma1.sim<-arima.sim(list(ma = c( -0.7)),
n = 100, sd=2);

plot.ts(ma1.sim, ylim=c(-6,8),main="MA(1), b= -0.7, n=100");
```

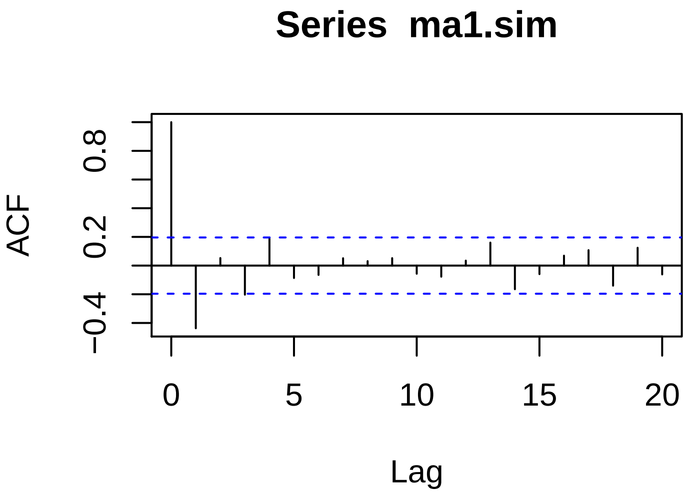
Scatterplot



Autocorrelation Function

```
acf(ma1.sim);
```

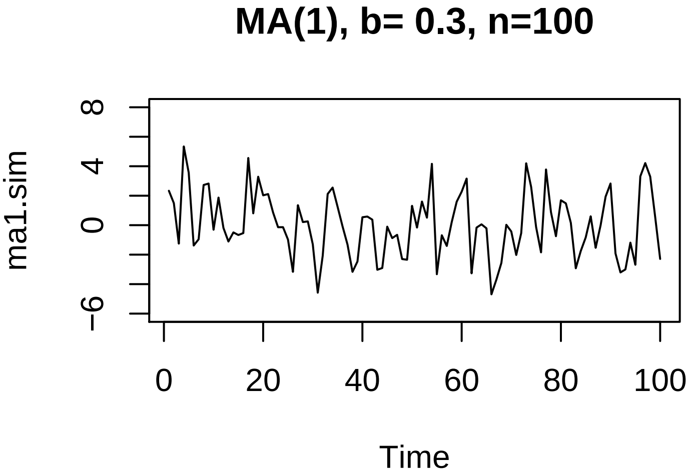
Autocorrelation Function, case i)



R Code

```
set.seed(9999);

# simulating MA(1);
ma1.sim<-arima.sim(list(ma = c(0.3)),
n = 100, sd=2);

plot.ts(ma1.sim, ylim=c(-6,8),main="MA(1), b= 0.3, n=100");
```

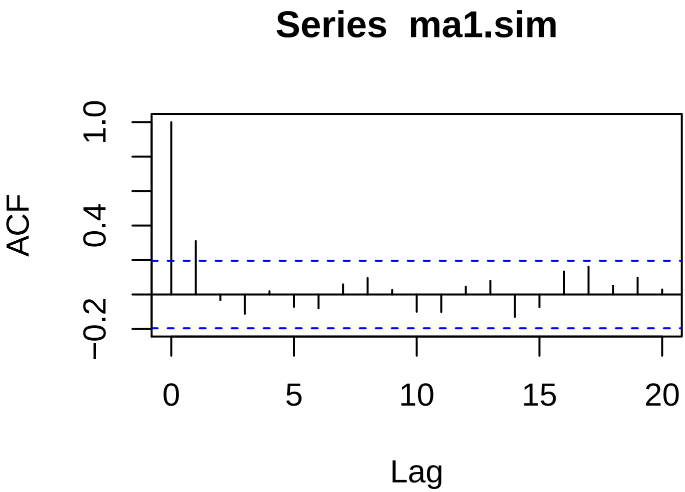
Scatterplot



Autocorrelation Function, case ii)

```
acf(ma1.sim);
```

Autocorrelation Function, case ii)



Simulated Examples of the MA(2) Model

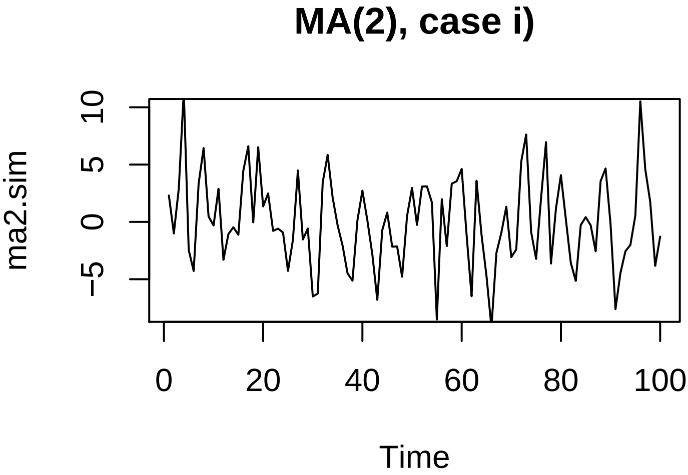
$$x_t = w_t + b_1 w_{t-1} + b_2 w_{t-2}.$$

Case i) $b_1 = 1.50$ and $b_2 = -0.56$
Case ii) $b_1 = 0.50$ and $b_2 = 0.24$
Case iii) $b_1 = -0.5$ and $b_2 = 0.24$
Case iv) $b_1 = 1.20$ and $b_2 = -0.72$

R Code

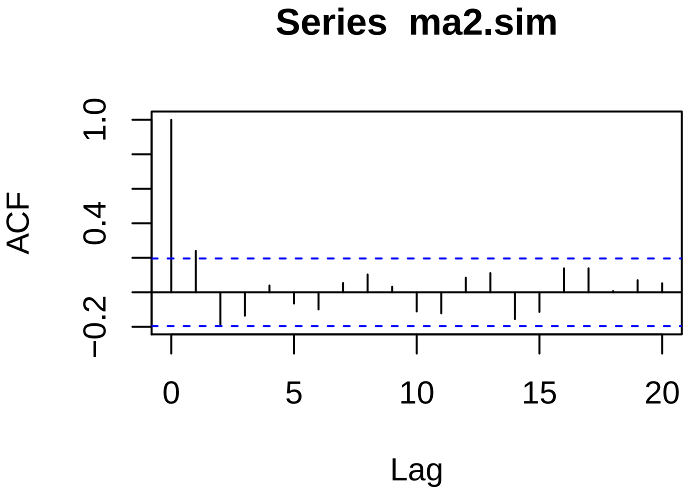
```r
b1<- 1.5;
b2<- -0.56;

set.seed(9999);

# simulating MA(2);
ma2.sim<-arima.sim(list(ma = c(b1,b2)),
n = 100, sd=2);


plot.ts(ma2.sim, ylim=c(-8,10),main="MA(2), case i)");
```

Scatterplot



Autocorrelation Function, case i)

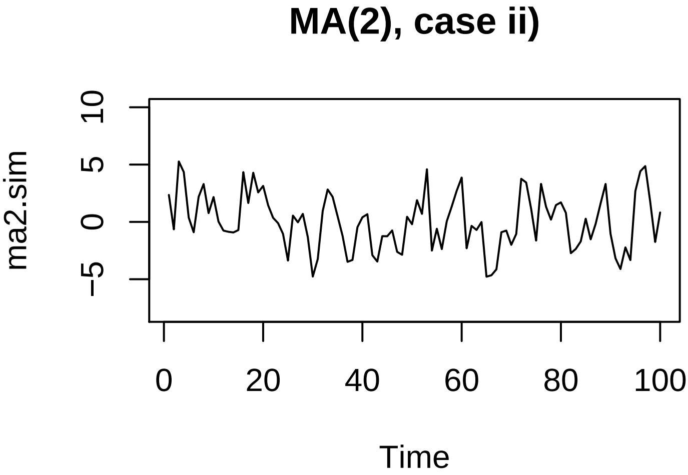
```
acf(ma2.sim);
```


Series ma2.sim
ACF
1.0
0.4
−0.2
0
5
10
15
20
Lag

R Code

```
b1<- 0.5;
b2<- 0.24;

set.seed(9999);

# simulating MA(2);
ma2.sim<-arima.sim(list(ma = c(b1,b2)),
n = 100, sd=2);


plot.ts(ma2.sim, ylim=c(-8,10),main="MA(2), case ii)");
```

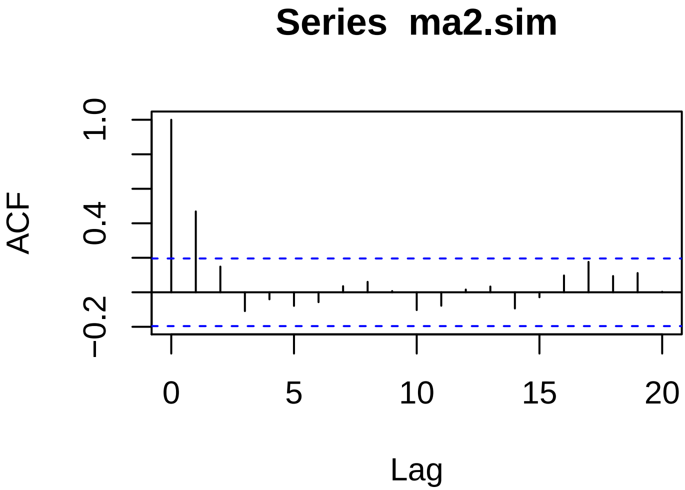
Scatterplot



Autocorrelation Function, case ii)

```
acf(ma2.sim);
```

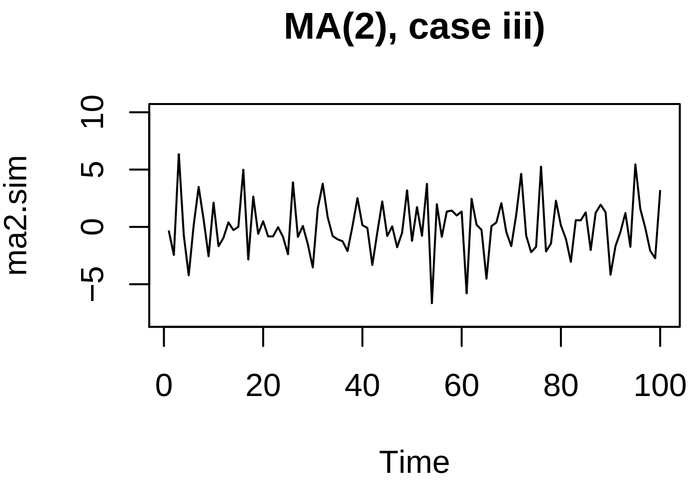
Autocorrelation Function, case ii)



R Code

```r
b1<- -0.5;
b2<- 0.24;

set.seed(9999);

# simulating MA(2);
ma2.sim<-arima.sim(list(ma = c(b1,b2)),
n = 100, sd=2);


plot.ts(ma2.sim, ylim=c(-8,10),main="MA(2), case ii)");
```

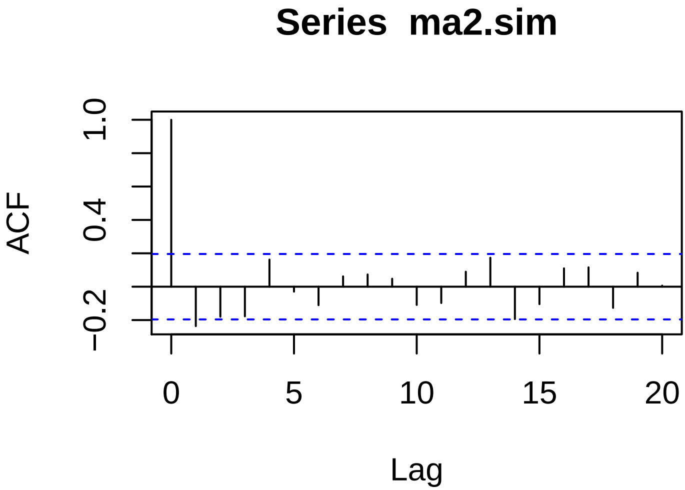
Scatterplot



Autocorrelation Function, case iii)

```
acf(ma2.sim);
```

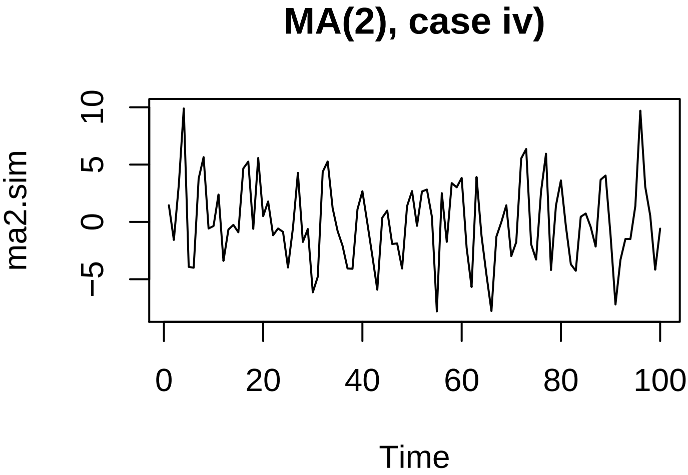
Autocorrelation Function, case iii)



R Code

```r
b1<- 1.20;
b2<- -0.72;

set.seed(9999);

# simulating MA(2);
ma2.sim<-arima.sim(list(ma = c(b1,b2)),
n = 100, sd=2);


plot.ts(ma2.sim, ylim=c(-8,10),main="MA(2), case ii)");
```

Scatterplot



Autocorrelation Function, case iv)

```
acf(ma2.sim);
```

Autocorrelation Function